PhD Qualifying Exam: Real Variables, Spring 2006

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Instructions: Solve any five problems. Credit will be given for the best five solutions.

Problem 1

Let $\langle f_n \rangle$ be a sequence of continuous, real-valued functions on a closed and bounded interval $I \subset \mathbb{R}$ such that for $k > l$,

$$f_k(x) \geq f_l(x) \quad \text{for all } x \in I.$$

Prove: If $\langle f_n \rangle$ converges pointwise to a continuous function f on I, then $\langle f_n \rangle$ converges uniformly on I.

Problem 2

(a) State the Hahn-Banach Theorem.

(b) Show that there exists a bounded linear functional $\Lambda : L^\infty(0,1) \to \mathbb{R}$ of norm 1 such that $\Lambda f = 0$ for all $f \in C([0,1])$.

(c) Show explicitly that there exists a bounded linear functional Λ on $L^\infty(0,1)$ for which no $g \in L^1(0,1)$ exists such that

$$\Lambda f = \int_{(0,1)} f\, g \quad \text{for every } f \in L^\infty(0,1).$$

Problem 3

(a) State Fatou's Lemma and the Lebesgue Monotone Convergence Theorem.

(b) Prove Fatou's Lemma using the Lebesgue Monotone Convergence Theorem.

Problem 4

(a) State and prove Jensen's inequality.

(b) Let $I \subset \mathbb{R}$ be an interval of finite, nonzero length and let $f \in L^1(\mathbb{R})$ be positive. Show that

$$m(I) \ln\left(\int_I f\right) \geq \int_I \ln f.$$

Problem 5

(a) State the Radon-Nikodym Theorem.

(b) Use the Radon-Nikodym Theorem to prove that any continuous, strictly increasing function $f : [a, b] \to \mathbb{R}$ which maps sets of measure zero to sets of measure zero is absolutely continuous.

Problem 6

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathcal{F}$ be the linear map that associates to each $f \in L^1(-\pi, \pi)$ the sequence $\mathcal{F}(f) = \langle \hat{f}(k) \rangle_{k \in \mathbb{N}_0}$ of real numbers, given by

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos(k\,x)\,dx.$$

(a) Prove the Riemann-Lebesgue Lemma: $\lim_{k\to\infty} \hat{f}(k) = 0$ for every $f \in L^1(-\pi, \pi)$, and conclude that $\mathcal{F}$ is a *bounded* linear operator from $L^1(-\pi, \pi)$ to c_0, the Banach space of all sequences $\langle a_k \rangle_{k \in \mathbb{N}_0}$ of real numbers such that $\lim_{k\to\infty} a_k = 0$, with the supremum norm

$$\|\langle a_k \rangle_{k \in \mathbb{N}_0}\|_\infty = \sup\{|a_k| \,:\, k \in \mathbb{N}_0\}.$$

(b) Show that $\mathcal{F} : L^1(-\pi, \pi) \to c_0$ is not one-to-one.

(c) For $n \in \mathbb{N}_0$ let the functions $D_n \in L^1(-\pi, \pi)$ be defined by

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right) x}{\sin \frac{x}{2}}.$$

Show that

$$\lim_{n\to\infty} \|D_n\|_1 = \infty$$

(Hint: $|\sin \frac{x}{2}| \le |\frac{x}{2}|$). Conclude, using the Open Mapping Theorem, that $\mathcal{F}$ is not onto. For this you may assume without proof that the restriction of $\mathcal{F}$ to the subspace of all *even* functions in $L^1(-\pi, \pi)$ is one-to-one and that

$$\sup_n \|\mathcal{F}(D_n)\|_\infty < \infty.$$

Problem 7

Suppose $f,\, g \in L^1(\mathbb{R})$. Show that

$$\int_{\mathbb{R}} |f(x - y)\, g(y)|\, dy < \infty \quad \text{a.e.}$$

and that the function

$$h(x) = \int_{\mathbb{R}} f(x - y)\, g(y)\, dy$$

is integrable and satisfies

$$\|h\|_1 \le \|f\|_1\, \|g\|_1.$$