ANALYSIS QUALIFYING EXAM,
SEPTEMBER 13, 2008

Do all 5 problems. Good luck.

1. (a) State carefully and precisely the Fundamental Theorem of Calculus for the Lebesgue integral.

(b) Let $f(x) = x^\theta \sin(1/x)$ for $0 \le x < 1$ and $f(0) = 0$. For which real values of θ is f absolutely continuous on $[0, 1]$?

2. Let $(\Omega, \sum, m\ )$ be a measure space. For $A_n \in \sum$ let

$$\limsup\ A_n = \{x \in \Omega : x \in A_n\ for\ infinitely\ many\ positive\ integers\ n\}\,.$$

(a) Show that $limsup\ A_n = \cap_{j=1}^{\infty} \cup_{k=j}^{\infty}\ A_k$ and conclude $limsup$ $A_n \in \sum$.

(b) Assuming $\sum_{n\ge 1} m(A_n) < \infty$, prove that $m(\ limsup\ A_n\ ) = 0$.

3. For $j = 1, 2$ let

$$f_j(t) = \int_0^\infty e^{-xt} g_j(x) dx$$

where g_j is continuous on $[0,\infty)$ and

$$|g_j(x)| \le 100 e^{\sqrt{x}}$$

for all positive x.

(a) Prove that f_1 is continuous on $(0, \infty)$.

(b) Prove that $\lim_{t\to\infty} f_1(t) = 0$.

(c) Give examples of g_1, g_2 so that $\lim_{t\to 0} f_1(t) = 5$, $\lim_{t\to 0} f_2(t) = -\infty$.

4. Let $A : Dom(A) \subset H \to H$ be a linear operator satisfying the condition

$$< Ax, y >=< x, Ay >$$

for all x, y in the domain $Dom(A)$; here $< .,. >$ is the inner product on a complex Hilbert space H. Call Φ_j an eigenvector of A corresponding to the eigenvalue b_j if Φ_j is a nonzero vector in $Dom(A)$ and $A\Phi_j = b_j\Phi_j$; here b_j is a complex number. Suppose that b_1 and b_2 are two different eigenvalues.

(a) Show that b_1 is real.

(b) Show that the corresponding eigenvectors satisfy $< \Phi_1, \Phi_2 >= 0$.

5. Consider two measures m_1, m_2 on $[0, \infty)$ equipped with its Borel sets; here m_1 is Lebesgue measure and m_2 has density e^{-x}. That is, for every Borel set E in $[0, \infty)$,

$$m_2(E) = \int_E e^{-x} dx.$$

Let M_j be the measure space $([0, \infty)$,Borel sets,$m_j)$. Is there any containment relationship between $L^1(M_j)$ and $L^2(M_j)$? That is, either prove that

$$L^1(M_1) \subset L^2(M_1) \quad \textit{or} \quad L^2(M_1) \subset L^1(M_1)$$

or give examples of functions in $L^2(M_1)\backslash L^1(M_1)$ and $L^1(M_1)\backslash L^2(M_1)$, and do the same thing with m_2 replacing m_1.