Qualifying Exam-Real Analysis (Fall 2010)

Do six of the following problems.

1. For any open interval I, $\ell(I)$ denotes the length of I. For any subset A of $\mathbb{R}$, the *outer measure* $m^*(A)$ of A is defined by

$$m^*(A) = \inf\{\sum \ell(I_n) : A \subseteq \cup I_n \ I_n\text{'s are countable open intervals}\}.$$

   Show that if $A \subseteq \cup_{n=1}^{\infty} A_n$, then $m^*(A) \leq \sum_{n=1}^{\infty} m^*(A_n)$.

2. Let f be an integrable function from $\mathbb{R}$ to $\mathbb{R}$. For each n, let h_n be the function defined by

$$h_n(t) = \frac{3 + t^{2n}}{1 + t^{2n}}.$$

   Show that $\lim_{n\to\infty} \int f(t)h_n(t)dt$ exists and find its limit.

3. Let f be an absolutely continuous function on a bounded interval $[a, b]$. Show that f is of bounded variation on $[a, b]$.

4. Let f be a nonnegative measurable function. Show that $\int f d\mu = 0$ implies $f = 0$ a.e.

5. Given an example of a sequence $\{f_n\}$ of measurable functions that converges to f in measure, but do not converges to f almost everywhere. Show that if $\{f_n\}$ is a sequence of measurable functions that converges to f in measure, then there is a subsequence of $\{f_n\}$ converges to f almost everywhere.

6. Let $\infty > p > 1$ and f is an L_p-function on $[0, 10]$. Show that f is integrable. Find the best constant c such that $\|f\|_1 \leq c\|f\|_p$ where $f \in L_p[0, 10]$. (Hint: Holder inequality)

7. Let f, g be two integrable functions on $\mathbb{R}$.

   (a) Show that for almost $x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} |f(x - y)g(y)|dy < \infty.$$

   (b) Let h be the function defined as

$$h(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy.$$

   Show that h is integrable and

$$\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1.$$

8. Recall that a subset A of L_1 is uniformly integrable if for any $\epsilon > 0$, there is $\delta > 0$ such that

$$\left| \int_E f dm \right| < \epsilon$$

whenever $f \in A$ and $m(E) < \delta$. Show that any finite subset of L_1 is uniformly integrable.