Real Analysis Qualifying Exam (spring 2010)

Do four of the following five problems.

1. (a) State carefully and precisely the Fundamental Theorem of Calculus for the Lebesgue integral.

   (b) Let

$$f(x) = \begin{cases} x^\theta \sin \frac{1}{x} & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0. \end{cases}$$

   For which real values of θ is f absolutely continuous on $[0, 1]$?

2. Let f be a Lebesgue integrable function from $\mathbb{R}$ to $\mathbb{R}$. The Fourier transform $\hat{f}$ of f is defined as

$$\hat{f}(t) = \int_{-\infty}^{\infty} e^{-ixt} f(x)dx \quad \text{for all } t \in \mathbb{R}.$$

   (a) Show that $\hat{f}$ is a continuous function on $\mathbb{R}$.

   (b) Suppose that $xf(x)$ is integrable, i.e., $\int_{-\infty}^{\infty} |xf(x)|dx < \infty$. Show that $\hat{f}$ is differentiable on $\mathbb{R}$, and

$$(\hat{f})'(t) = \int_{-\infty}^{\infty} (-ix)e^{-ixt} f(x)dx.$$

3. Suppose that $1 \leq s < t < \infty$. Let λ be the Lebesgue measure on $[0, 2]$. Show that there is a constant c such that for any function f in $L^t([0, 2])$,

$$\|f\|_s \leq c\|f\|_t.$$

   What is the best constant?

4. Let f, g be two integrable functions on $\mathbb{R}$.

   (a) Show that for almost $x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} |f(x - y)g(y)|dy < \infty.$$

   (b) Let h be the function defined as

$$h(x) = \int_{-\infty}^{\infty} f(x - y)g(y)dy.$$

   Show that h is integrable and

$$\|h\|_1 \leq \|f\|_1 \cdot \|g\|_1.$$

5. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space and f an element in $L^1(\Omega, \mathcal{B}, \mu)$. Let ν be the function from $\mathcal{B}$ to $\mathbb{R}$ defined by

$$\nu(E) = \int_E f d\mu.$$

Show that ν is a signed measure. What is $|\nu|$?