Ph.D. Qualifying Exam, September 2011
Real Variables

In order to obtain full credit, solve five out of the following seven problems. Please write carefully and add sufficient explanations.

Problem 1

(a) State the Lebesgue Dominated Convergence Theorem.

(b) Let $f : \mathbb{R} \to [-\infty, \infty]$ be a Lebesgue integrable function. For $n \in \mathbb{N}$, let $h_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$h_n(x) = \frac{3\, e^{n\,x}}{2 + e^{n\,x}}.$$

Show that the sequence $\left(\int (h_n\, f) \right)$ is convergent. Determine its limit.

Problem 2

Let m^* denote the Lebesgue outer measure on $\mathbb{R}$. Recall:

- A set $F \subset \mathbb{R}$ is called an F_σ-set if F is the union of a countable collection of closed sets.
- A function $f : [a, b] \to \mathbb{R}$ on a compact interval $[a, b]$ is called Lipschitz continuous if there exists a constant $L > 0$ such that $|f(x) - f(y)| \leq L\, |x - y|$ for all $x,\ y \in [a, b]$.

(a) Let $S \subset \mathbb{R}$ be a set. Prove: If there exists an F_σ-set F contained in S such that $m^*(S \setminus F) = 0$, then S is Lebesgue measurable.

(b) Let $f : [a, b] \to \mathbb{R}$ be Lipschitz continuous. Show that f maps F_σ-sets in $[a, b]$ onto F_σ-sets, sets of measure 0 in $[a, b]$ onto sets of measure 0, and Lebesgue measurable sets in $[a, b]$ onto Lebesgue measurable sets.

Problem 3

Let m be the Lebesgue measure on $\mathbb{R}$. Suppose $\{S_k \,|\, k \in \mathbb{N}\}$ is a countable collection of measurable sets in $\mathbb{R}$ such that $\sum_{k=1}^{\infty} m(S_k) < \infty$. Prove: The set of points $x \in \mathbb{R}$ which belong to at least one infinite subcollection of $\{S_k \,|\, k \in \mathbb{N}\}$ has measure 0.

Problem 4

Let $\mathcal{M}$ be the σ-algebra of all sets E in $[0, 1]$ such that either E or $[0, 1] \setminus E$ is countable. Let μ be the counting measure on $\mathcal{M}$.

(a) Show that the function $g(x) = x$, $0 \leq x \leq 1$ is not measurable.

(b) Show that for each $f \in L^1([0, 1], \mathcal{M}, \mu)$, $f\, g$ is integrable. Prove that the map $f \mapsto \int f\, g\, d\mu$ defines a bounded linear functional on $L^1([0, 1], \mathcal{M}, \mu)$.

(c) Conclude that the dual space of $L^1([0, 1], \mathcal{M}, \mu)$ is not isometrically isomporhic to $L^\infty([0, 1], \mathcal{M}, \mu)$. How does this result relate to the Riesz Representation Theorem?

TURN!

Problem 5

(a) State the Hahn-Banach Theorem.

(b) Let X be a Banach space. Show that for every $x \in X$, there exists a bounded linear functional f on X such that $f(x) = ||f||\,||x||$.

Problem 6

Let X be $C([0,1])$ endowed with the maximum norm $\|f\|_\infty = \max\{|f(x)|\,|\,0 \leq x \leq 1\}$ and let Y be $C([0,1])$ endowed with the L^1-norm $\|f\|_1 = \int_{[0,1]} |f|$, $f \in C([0,1])$. Let $I : X \to Y$ be the identity operator from X to Y. Prove that I maps the open unit ball in X to a set which is not open in Y. Use this result to conclude that Y is not a Banach space.

Problem 7

Let

$$f(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Is f Lebesgue integrable over $\mathbb{R}^2$? Argue carefully.