Qualifying Exam (2011 spring)-Real Analysis

Do five of the following seven problems.

Problem 1

Let g be an integrable function and suppose $\lim_{x\to 1} g(x) = 0$. Prove that

$$\lim_{n\to\infty} \int_{[0,1]} n\, x^n\, g(x)\, dx = 0.$$

Problem 2

Let f be an integrable function. Prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, \sin(nx)\, dx = 0.$$

Hint: Prove this first when f is the characteristic function of an interval.

Problem 3

Let $S \subset \mathbb{R}$ be measurable such that $0 < m(S) < \infty$. Prove that

$$\lim_{h\to 0} \frac{m(S \cap (x - |h|, x + |h|))}{2|h|} = \chi_S(x)$$

for almost every $x \in \mathbb{R}$.

Problem 4

Let $\{S_n \,|\, n \in \mathbb{N}\}$ be a collection of measurable sets such that $S_n \subset S_{n+1}$ and let m be Lebesgue measure on $\mathbb{R}$. Prove that

$$\lim_n m(S_n) = m\left(\bigcup_{n\in\mathbb{N}} S_n\right).$$

Problem 5

Let $(X, \mathcal{B}, \mu)$ be a finite (positive) measure. Call a set $\Phi \subseteq L_1(\mu)$ *uniformly integrable* if to each $\epsilon > 0$ corresponds a $\delta > 0$ such that

$$\left|\int_E f d\mu\right| < \epsilon$$

whenever $f \in \Phi$ and $\mu(E) < \delta$. Show that every finite subset of $L_1(\mu)$ is uniformly integrable.

Problem 6

Suppose that $1 \le p < \infty$, $f \in L_1(\mathbb{R})$, $g \in L_1(\mathbb{R}) \cap L_p(\mathbb{R})$. $f * g$ is defined by

$$f * g(y) = \int f(y - x) g(x) dx.$$

Show that

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

Problem 7

A function f is said to satisfy a Lipschitz condition on an interval if there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all x and y in the interval.

1. Show that a function satisfying a Lipschitz condition is absolutely continuous.
2. Show that an absolutely continuous function f on a finite interval $[a, b]$ satisfies a Lipschitz condition if and only if $|f'|$ is essentially bounded.