Ph.D. Qualifying Exam
Real Variables, August 2013

Solve any 5 of the following 8 problems. Please write carefully and give sufficient explanations.

Problem 1

Let $A_n \subset \mathbb{R}$, $n \in \mathbb{N}$. Define

$$\underline{\lim} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n, \quad \overline{\lim} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n.$$

Let m be the Lebesgue measure on $\mathbb{R}$ and m^* its outer measure.

(a) Show that for any sequence of Lebesgue measurable sets A_n it holds

$$m(\underline{\lim} A_n) \leq \underline{\lim}\ m(A_n).$$

(b) Show that for any sequence of sets $A_n \subset \mathbb{R}$,

$$m^*(\underline{\lim} A_n) \leq \underline{\lim}\ m^*(A_n).$$

Problem 2

Let $f_n : A \to \mathbb{R}$, $A \in \mathcal{M}$, be measurable and $f_n \geq 0$. Show:

(a) If $\lim_{n\to\infty} \int_A f_n = 0$ then $f_n \to 0$ in measure on A.

(b) Give an example that the measure convergence cannot be replaced by convergence a.e.

Problem 3

Prove or disprove:

$$\lim_{n\to\infty} \int_0^1 n^2 x(1-x)^n \, dx = \int_0^1 \lim_{n\to\infty} n^2 x(1-x)^n \, dx.$$

Problem 4

Let $f_n : \mathbb{R} \to \mathbb{R}$ be nonnegative and Lebesgue integrable functions on $\mathbb{R}$ such that f_n is convergent to f on $\mathbb{R}$. Assume also that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dx = \int_{\mathbb{R}} f(x)\, dx < \infty$. Show that for each Lebesgue measurable set $A \subset \mathbb{R}$,

$$\lim_{n\to\infty} \int_A f_n(x) dx = \int_A f(x)\, dx.$$

HINT: Apply the Fatou Lemma.

TURN!

Problem 5

State and prove the Minkowski Inequality in L^p for $1 \leq p < \infty$.

Problem 6

(I) State the Radon-Nikodym Theorem for σ-finite measure space $(X, \mathcal{B}, \mu)$.

(II) Let μ, ν, ν_i, $i = 1, 2$, be σ-finite measures on the measurable space $(X, \mathcal{B})$. Let the symbol $\left[\frac{d\nu}{d\mu}\right]$ denote the Radon-Nikodym derivative of ν with respect to μ. Show:

(i) If ν is absolutely continuous with respect to μ, that is $\nu \ll \mu$, and f is a nonnegative measurable function, then

$$\int f \, d\nu = \int f \left[\frac{d\nu}{d\mu}\right] \, d\nu.$$

(ii) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$ then

$$\left[\frac{d(\nu_1 + \nu_2)}{d\mu}\right] = \left[\frac{d\nu_1}{d\mu}\right] + \left[\frac{d\nu_2}{d\mu}\right].$$

Problem 7

Recall that the space ℓ^∞ consists of all sequences $x = (\xi_j)$ such that $\|x\|_\infty = \sup_{j \in \mathbb{N}} |\xi_j| < \infty$.

(a) Show that $T : \ell^\infty \to \ell^\infty$ defined by $y = (\eta_j) = Tx$, $\eta_j = \frac{\xi_j}{j}$ for $x = (\xi_j)$, is linear and bounded.

(b) Let $\mathcal{R}(T)$ be the range of T. Show that $\mathcal{R}(T)$ is not a closed subspace of ℓ^∞.

(c) Consider the inverse operator $T^{-1} : \mathcal{R}(T) \to \ell^\infty$, $\mathcal{R}(T) \subset \ell^\infty$. Show that T^{-1} is unbounded.

Problem 8

Let $E = [0, 1] \times [0, 1]$, m^2 be the product Lebesgue measure on $\mathbb{R}^2$, and

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Investigate the existence and equality of

$$\int_E f dm^2, \quad \int_0^1 \int_0^1 f(x, y) \, dx dy \quad \text{and} \quad \int_0^1 \int_0^1 f(x, y) \, dy dx.$$