Ph.D. Qualifying Exam
Real Variables, January 2013

Solve any 5 of the following 7 problems. Please write carefully and give sufficient explanations.

Problem 1

Let m denote Lebesgue measure on $\mathbb{R}$. Recall that for a measurable set $E \subset \mathbb{R}$ with $m(E) > 0$ and $\epsilon > 0$, there is an interval $I \subset \mathbb{R}$ such that $m(I \cap E) > (1 - \epsilon)\, m(I)$.

Prove: If $E, F \subset \mathbb{R}$ are measurable with $m(E) > 0$ and $m(F) > 0$, then the set $E - F = \{x - y \,|\, x \in E, y \in F\}$ contains a nontrivial interval.

Problem 2

Let

$$f(x) = \begin{cases} x^2 |\sin\left(\frac{1}{x}\right)| & \text{for } \; x \in (0, 1]; \\ 0 \;\; \text{for } \; x = 0, \end{cases}$$

and let

$$g(x) = \sqrt{x}.$$

(a) Show that f and g are absolutely continuous on $[0, 1]$.

(b) Show that the composition $f \circ g$ is absolutely continuous, but $g \circ f$ is not absolutely continuous on $[0, 1]$.

Problem 3

Let $\mathfrak{M}$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. Let $f_n : A \to \mathbb{R}$ be measurable functions on $A \in \mathfrak{M}$ with $m(A) < \infty$. Show that

$$\lim_{n\to\infty} \int_A \frac{|f_n|}{1 + |f_n|} = 0 \;\Leftrightarrow\; f_n \to 0 \; \text{ in measure on } A.$$

HINT: Prove first that the function $g(x) = \frac{x}{1+x}$ is increasing on $[0, \infty)$.

Problem 4

Let $A \in \mathfrak{M}$ with $m(A) < \infty$ and $0 < p_1 < p_2 < \infty$. Show that $L^{p_2}(A) \subset L^{p_1}(A)$ and for any $f \in L^{p_2}(A)$ it holds $\|f\|_{p_1} \leq \|f\|_{p_2} (m(A))^{\frac{1}{p_1} - \frac{1}{p_2}}$.

Recall that $L^p(A)$, $0 < p < \infty$, is the set of all Lebesgue measurable functions f on A such that $\|f\|_p = \left(\int_A |f|^p\right)^{1/p} < \infty$.

TURN!

Problem 5

Let

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & 0 < x < 1; \\ 0, & \text{otherwise on } \mathbb{R}. \end{cases}$$

Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of all rational numbers and $g(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n)$.

(a) Show $\int_{\mathbb{R}} g < \infty$.

(b) Show that g is not continuous at any $x \in \mathbb{R}$.

(c) Show $\int_{\mathbb{R}} g^2 = \infty$.

Problem 6

Prove: A linear functional f on a normed linear space X is bounded if and only if the kernel of f,

$$\ker f = \{x \in X \mid f(x) = 0\},$$

is closed.

Problem 7

Let $(X, \mathcal{M}, \mu)$ be a finite measure space, where X is an abstract set, $\mathcal{M}$ is a σ-algebra of subsets of X and μ is a measure on $\mathcal{M}$. Let $g \geq 0$ be integrable and $f \geq 0$ measurable functions. Let further $\nu : \mathcal{M} \to [0, \infty]$ be defined for $E \in \mathcal{M}$ by

$$\nu(E) = \int_E g \, d\mu.$$

(a) Show that ν is a finite measure. If μ is complete, is ν as well?

(b) Show that for every $E \in \mathcal{M}$,

$$\int_E f \, d\nu = \int_E fg \, d\mu.$$