Qualifying Exam

Real Variables

Solve four out of the following seven problems.

Problem 1. a) If X is an infinite dimensional normed linear space, prove that O is in the weak closure of S_X.
b) Let X be a Banach space and $x, x_1, x_2, \cdots, x_n, \cdots \in X$. If x_n converges weakly to x then there exists a sequence $\{y_n\}$ such that each y_n is a convex combination of elements from the sequence $\{x_n\}$ and $\{y_n\}$ converges strongly to x.
c) Show that the weak topology in ℓ_2 is not metrizable.

Problem 2. Let $(V, \|\cdot\|)$ be a nontrivial real normed vector space. Let W be a linear and closed subspace of V.
a) Use the Hahn Banach theorem to prove that for every real $\epsilon > 0$, there exists $v_\epsilon \in V$ with $\|v_\epsilon\| = 1$ and $\|v_\epsilon - w\| \geq 1 - \epsilon$, for all $w \in W$.
b) If the unit ball of V is compact prove that there exists $v_0 \in V$ of norm one such that $\|v_0 - w\| \geq 1$, for all $w \in W$.

Problem 3. a) If f is a continuous and convex function on $[a, b]$, show that

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f.$$

b) If f is a continuous and nonnegative function on $[a, b]$ with $p \geq 1$, prove that

$$\left(\frac{1}{b-a}\int_a^b g\right)^p \leq \frac{1}{b-a}\int_a^b g^p.$$

Problem 4. Let L^1 and L^2 be the usual Lebesgue spaces over the interval $[0, 1]$. Prove the following statements:
a) $\{f : \int |f|^2 \leq 1\}$ is closed in L^1 and has empty interior.
b) If $g_n = \chi_{[0,n^{-3}]}\, n$, then $\int f g_n \to 0$ for every $f \in L^2$ but not for every $f \in L^1$.
c) The inclusion map L^2 into L^1 is continuous but not onto.

Problem 4. Let ν, μ and λ be finite measures on the measurable space $(X, \mathcal{A})$. Prove the following statements:
a) If $\nu \ll \mu$ and $\lambda \ll \mu$ then

$$\frac{d(\nu+\lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu}, \quad a.e.[\mu]$$

b) If $\nu \ll \mu \ll \lambda$ then

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda} \quad a.e.[\lambda]$$

Problem 5. Let E be a Hilbert space. Let M and N be closed subspaces of E. Prove that:
a) $M^{\perp\perp} = M$.
b) If M and N are orthogonal then $M + N$ is a closed subspace of E.

c) If M and N are not orthogonal then $M+N$ is not necessarily closed.

Problem 6. a) Let X and Y be two given Banach spaces. Suppose T and T_n $(n = 1, 2, \cdots)$ are bounded operators from X into Y. If each T_n has finite dimensional range and $\lim_n \|T_n - T\| = 0$ then T is compact.
b) If Y is a Hilbert space and T is compact, construct a sequence of operators $\{T_n\}_n$, such that each T_n has finite dimensional range and $\lim_n \|T_n - T\| = 0$.

Problem 7. Prove the following statements: Let X and Y be Banach spaces.
a) If $\{x_n\}_n$ is a weakly convergent sequence in X then $\{\|x_n\|\}$ is bounded.
b) If $T : X \to Y$ is a bounded operator and $x_n \to x$ weakly then $Tx_n \to Tx$ weakly.
c) If $T : X \to Y$ is a bounded compact operator and $x_n \to x$ weakly then $\|Tx_n - Tx\| \to 0$.