Ph.D. Qualifying Exam
Real Variables

Solve any four of the following six problems. Please write carefully and give sufficient explanations.

Problem 1

For $1 \le p \le \infty$, let $l^p = L^p(\mathbb{N}, \mu)$ where μ denotes the counting measure. Hence l^p consists of sequences (x_n) such that $\|(x_n)\|_p = (\sum_{k=1}^{\infty} |x_n|^p)^{1/p} < \infty$ in case $1 \le p < \infty$ and $\|(x_n)\|_\infty = \sup\{|x_n| \mid n \in \mathbb{N}\} < \infty$ in case $p = \infty$.

Let c_0 be the closed subspace of l^∞ consisting of all sequences that converge to 0.

(a) Prove: If (y_n) is a sequence in l^1 and f is defined for (x_n) in c_0 by $f((x_n)) = \sum_1^\infty x_k\, y_k$, then f is a bounded linear functional on c_0 and $\|f\| = \|(y_n)\|_1$.

(b) Conclude that c_0^*, the dual space of c_0, and l^1 are isometrically isomorphic.

(c) Prove: Every sequence (y_n) in l^1 gives rise to a bounded linear functional on l^∞ as in part (a). However, there is a non-zero bounded linear functional on l^∞ that vanishes on all of c_0.

Problem 2

Let $\{e_k \mid k \in \mathbb{N}\}$ and $\{f_k \mid k \in \mathbb{N}\}$ be two orthonormal bases of $L^2(0,1)$. Let $X = \operatorname{span}\{e_k \mid k \in \mathbb{N}\}$, endowed with the L^2-norm. Hence X consists of *finite* linear combinations of the basis elements e_k. Consider the linear operator $T : X \to L^2(0,1)$, defined by

$$Te_k = k\, f_k, \quad k \in \mathbb{N}.$$

Show that T is an unbounded, injective operator, that its inverse is bounded and that the range of T is dense in $L^2(0,1)$.

Problem 3

Let (f_n) be a sequence of real valued, integrable functions defined on a measurable subset of $\mathbb{R}$, denoted by E. Prove that, if $\sum_{n=1}^\infty \int_E |f_n| < \infty$ then $\sum_{n=1}^\infty f_n$ converges almost everywhere on E to an integrable function f and $\int_E f = \sum_{n=1}^\infty \int_E f_n$.

Problem 4

Let $[a,b]$ be a closed interval in $\mathbb{R}$. Denote by $\mathcal{BV}_0$ the set of all functions of bounded variation on $[a,b]$ that vanish at a. For each function f on $[a,b]$, set $\|f\| = V_a^b f$ where V_a^b denotes total variation on $[a,b]$. Prove:

1. The product of two functions in $\mathcal{BV}_0$ is in $\mathcal{BV}_0$.
2. $\|f\| = V_a^b f$ defines a norm on $\mathcal{BV}_0$.
3. A Cauchy sequence of functions in $\mathcal{BV}_0$ converges to a function in $\mathcal{BV}_0$.

Problem 5

Prove:

(a) If f is an integrable function defined on a measurable set $E \subset \mathbb{R}$, then the set

$$A = \{x \in E : f(x) \neq 0\}$$

is σ-finite (i.e. it can be written as a countable union of measurable sets, each with finite measure).

(b) Suppose f and g are measurable functions on $\mathbb{R}$. If $\sqrt{f^2 + g^2}$ is integrable then $f\,g$ is integrable.

Problem 6

(a) Define what it means for a real function to be absolutely continuous on a closed interval $[a, b] \subset \mathbb{R}$.

(b) Let f be a continuous function on $[a, b]$ of bounded variation such that f is absolutely continuous on $[a, c]$ for all $c \in (a, b)$. Prove that f is absolutely continuous on $[a, b]$.

(c) Let f be defined on $[0, 1]$ by $f(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $f(0) = 0$. Prove that f has finite derivative at every point. Is f absolutely continuous? Explain your answer.