Qualifying Examination

Analysis

August 19, 2015

Complete any five of the following seven problems. You have three hours.

(1) Let L^1 and L^2 be the usual Lebesgue spaces over the interval $[0, 1]$. Prove the following statements:
a) $\{f : \int |f|^2 \leq 1\}$ is closed in L^1 and has empty interior.
b) The inclusion map L^2 into L^1 is continuous but not onto.

(2) (a) Use the Hahn-Banach Theorem for real normed spaces to prove the Hahn-Banach Theorem for complex normed spaces.
(b) Show that if Y is a closed subspace of a normed space X and $x \in X$ but $\notin Y$ then there exists a bounded linear functional f on X such that $\|f\| = 1$, $f(y) = 0$ for all $y \in Y$ and $f(x) = d(x, Y)$.

(3) Let $f : [a, b] \to \mathbb{R}$.
(a) Prove that if f is a Lipschitz function then f is absolutely continuous.
(b) State precisely the Fundamental Theorem of Calculus for the Lebesgue integral.

(4) Prove the following statements:
(a) Every finite dimensional normed space is reflexive.
(b) Every bounded linear functional on a reflexive normed space X is norm attaining.

(5) Consider the operator L given by

$$Lf(x) = \int_0^\infty exp(-tx)f(t)dt.$$

View L as a linear operator on the space $L^p(0, \infty)$. Show that L is unbounded if $1 < p < \infty$ and $p \neq 2$. **Hint**: consider the functions $f_r(x) = exp(-rx)$, for $r > 0$.

(6) Let $\{f_n\}$ be a sequence in $L^1(0, \infty)$. If $f_n(x) \to f(x)$ for a.e. in $(0, \infty)$ does

$$\int_0^\infty |f_n(x) - f(x)|dx \to 0?$$

Give a proof or find a counterexample. Is the function f necessarily integrable? Explain your answer.

(7) State the Radon-Nikodym Theorem for σ-finite measures. Let ν, μ and λ be σ-finite measures on the measurable space $(X, \mathcal{A})$.
a) Assume that $\nu \ll \mu$ and $\lambda \ll \mu$. Show that

$$\frac{d(\nu + \lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu}, \ \ a.e.[\mu].$$

b) If $X = [0, 1]$ and $\mathcal{M}$ is the collection of Lebesgue measurable subsets of X and take ν to be the Lebesgue measure and μ the counting measure on $\mathcal{M}$. Show that ν is absolutely continuous with respect to μ but there is no function f for which $\nu(A) = \int_A f \, d\mu$ for all $A \in \mathcal{M}$. Explain why this is not a counterexample to the Radon-Nikodym Theorem.