PhD Qualifying Exam in Real Analysis
January 2015

Do 6 of the 8 problems

1. Let $f : [a, b] \to \mathbb{R}$.
(a) What does it mean for f to be absolutely continuous?
(b) Let f be absolutely continuous and satisfy $f(x) > \varepsilon > 0$ for all $x \in [a, b]$. Prove that $g = 1/f$ is absolutely continuous.

2. State Hölder's inequality for functions on the unit interval [0,1]. Prove that if $1 \le p < r < \infty$, then $f \in L^r[0, 1]$ implies $f \in L^p[0, 1]$.

3. Let (Ω, Σ, μ) be a measure space.
(a) State Fatou's Lemma for a sequence of $\Sigma-$measurable functions.
(b) Use Fatou's Lemma to prove Lebesgue's Dominated Convergence Theorem, which you should state precisely.

4. Let X be a Hilbert space. Let $\{f_n\}$ be a sequence in X.
(a) Suppose f_n converges weakly to f. Prove that $\|f_n\|$ is bounded.
(b) Suppose that in (a), we assume in addition that $\|f_n\| \to \|f\|$. Then prove that $\|f_n - f\| \to 0$.

5. Prove or disprove the following statement. If $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of Lebesgue integrable functions and $f_n \to 0$ in measure, then $f_n \to 0$ in $L^1(\mathbb{R})$.

6. Let $E \subset [0, 1]$ have positive Lebesgue outer measure, and let $0 < a < 1$ be given. Prove that there is an interval L such that the Lebesgue outer measure of $E \backslash L$ is at least a times the length of L.

7. Show that any normed vector space can be isometrically embedded into a Banach space.

8. Let $f \in L^1(0, \infty)$. Define

$$g(t) = \int_0^\infty e^{-tx} f(x) dx.$$

Prove that g is bounded and continuous on $[0, \infty)$ and

$$\lim_{t \to \infty} g(t) = 0.$$