Real Analysis Qualifying Exam

August, 2016

Please solve 4 of the following 7 problems, employing clarity of exposition and logical methodology.

(1) Let $(X, \mathcal{B}, \mu)$ be a probability space (i.e. μ is a positive measure on the sigma-algebra $\mathcal{B}$ with $\mu(X) = 1$). Recall that a sequence of real-valued measurable functions $\{f_n\}_{n\in\mathbb{N}}$ is said to *converge in measure* to a measurable function f iff for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for every $n > N$,

$$\mu\{x : |f_n(x) - f(x)| > \epsilon\} < \epsilon.$$

(a) Show that if $f_n \to f$ μ-almost everywhere, then $f_n \to f$ in measure. Give an example to show that the converse is false.

(b) Suppose each $f_n \in L^1(\mu)$ and $f_n \to f$ in L^1. Show that $f_n \to f$ in measure. Is the converse true? Prove or give a counterexample.

(2) Let $(X, \mathcal{B}, \mu)$ be a probability space (i.e. μ is a positive measure on the sigma-algebra $\mathcal{B}$ with $\mu(X) = 1$).

(a) Define the *essential supremum* of a measurable function $f : X \to \mathbb{R}$.

(b) Define the space $L^\infty(X, \mathcal{B}, \mu)$, and describe the usual norm.

(c) Show that $f_n \to f$ in $L^\infty(X, \mathcal{B}, \mu)$ iff there exists a set $E \subset X$ with $\mu(E) = 0$ so that $f_n \to f$ uniformly off of E.

(3) Recall that a Banach space is **separable** if there exists a countable, dense (in the norm topology) subset. Let ℓ^1 be the usual space of all absolutely summable real sequences, and ℓ^∞ the space of bounded real sequences, each equipped with their usual norms.

(a) Show directly that ℓ^1 is separable.

(b) Show directly that ℓ^∞ is not separable.

(4) Let $I = [0, 1]$ denote the unit interval in $\mathbb{R}$.

(a) Define the *Cantor function* $f : I \to I$.

(b) Show that f is continuous and of bounded variation, but not absolutely continuous.

(5) Let X be a Banach space with $\mathcal{L}(X)$ the space of bounded operators on X, and let $I \in \mathcal{L}(X)$ denote the identity operator ($I(x) = x\ \forall x \in X$). Show the following:

(a) If $T \in \mathcal{L}(X)$ and $\|I - T\| < 1$ then T is invertible. Hint: consider an appropriate series.

(b) If $T \in \mathcal{L}(X)$ is invertible and $\|S - T\| < \|T^{-1}\|^{-1}$ then S is invertible.

Conclude that the set of invertible operators on X is open.

(6) Suppose that $f \geq 0$ is a (Lebesgue) measurable function on $\mathbb{R}$. Let

$$E_k = \{x : f(x) > 2^k\}, \quad \text{and} \quad F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}.$$

Show that

$$f \text{ is integrable} \quad \text{iff} \quad \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty \quad \text{iff} \quad \sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty.$$

(7) Let $(X, \|\cdot\|)$ be a (real) Banach space and $\tau : X \to \mathbb{R}$ a linear functional on X. Show that τ is bounded if and only if the kernel of τ,

$$\ker \tau = \{x \in X : \tau(x) = 0\},$$

is closed.