Real Analysis Qualifying Exam

Fall 2023

Credit will be given for the best five questions; Show all steps and arguments and state clearly all theorems that you apply.

1. Let S be a dense subset of $\mathbb{R}$. Show that if f is an extended real-valued function defined on $\mathbb{R}$ such that $\{x : f(x) < \alpha\}$ is measurable for every $\alpha \in S$, then f is measurable.

2. Prove or disprove the following statement. If $f_n : (0,1) \to \mathbb{R}$ is a sequence of Lebesgue integrable functions and $f_n \to 0$ in measure, then $f_n \to 0$ in $L^1(0,1)$.

3. Let ν be a signed measure and μ be a measure. State the definition of ν to be absolutely continuous with respect to μ. Prove that if $f \in L^1(\mu)$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that $\int_E f d\mu < \epsilon$ whenever $\mu(E) < \delta$.

4. Let ν, μ and λ be σ-finite measures on the measurable space $(X, \mathcal{A})$.
   a) If $\nu \ll \mu$ and $\lambda \ll \mu$, then $d(\nu + \lambda)/d\mu = d\nu/d\mu + d\lambda/d\mu$, μ-a.e.
   b) if $X = [0,1]$ and $\mathcal{M}$ is the collection of Lebesgue measurable subsets of X and take ν to be the Lebesgue measure and μ the counting measure on $\mathcal{M}$. Show that $\nu \ll \mu$ but there is no function f for which $\nu(A) = \int_A f d\mu$ for all $A \in \mathcal{M}$.

5. Show that $L^p(0,1)$ $(1 \le p < \infty)$ is a Banach space and separable.

6. State Hölder's inequality for functions on the unit interval $(0,1)$. Prove that if $1 \le p < q < \infty$, then $L^q(0,1) \subset L^p(0,1)$.

7. State the Closed Graph Theorem. Let X and Y be Banach spaces. Prove that if a linear operator $T : X \to Y$ satisfies $f \circ T \in X^*$ for every $f \in Y^*$, then T is bounded.

8. Let (x_n) be a sequence in a Banach space X. Prove that if $x_n \to x$ weakly, then (x_n) is bounded and $\|x\| \le \liminf_n \|x_n\|$.