PH.D. QUALIFYING EXAM: REAL ANALYSIS

Fall 2024

Solve six of the eight problems. Show all steps and formulate theorems cited.

1. Suppose that $f, g \in L^1(X)$, where X is a probability space. Suppose also that $|f(x)g(x)| \geq c > 0$ for all $x \in X$. Show that

$$\left(\int_X |f| \right) \left(\int_X |g| \right) \geq c.$$

2. Let $(X, \mathcal{B}, \mu)$ be a measure space and suppose that f is real valued and μ-integrable on X. Show that $\{x \in X : f(x) \neq 0\}$ is μ σ-finite, i.e. a countable union of finite μ-measure sets.

3. Let $(X, \mathcal{B}, \mu)$ be a measure space.

(a) Prove the continuity from below property. Namely, if (E_n) is a sequence of measurable sets with $E_1 \subset E_2 \subset E_3 \subset \cdots$ then $\mu(\bigcup_1^\infty E_n) = \lim_n \mu(E_n)$.

(b) Prove the continuity from above property. Namely, if (E_n) is a sequence of measurable sets with $E_1 \supset E_2 \supset E_3 \supset \cdots$ and $\mu(E_1) < \infty$ then $\mu(\bigcap_1^\infty E_n) = \lim_n \mu(E_n)$.

4. Let $(X, \mathcal{B}, \mu)$ be a measure space and suppose that $(A_n)_{n=1}^\infty$ is a sequence of measurable subsets of X. Define

$$\liminf_n A_n = \{x \in X : |\{k \in \mathbf{N} : x \notin A_k\}| < \infty\}.$$

(a) Prove that $\liminf_n A_n \in \mathcal{B}$. (b) Prove that if $A_1 \supset A_2 \supset A_3 \supset \cdots$ and

$$\infty > \lim_{n\to\infty} \mu(A_n) = c > 0$$

then $\mu(\liminf_n A_n) = c$.

5. Let $X = [1, \infty)$. Define two Borel measures on X by, for $E \subset X$ Borel measurable and $m =$ Lebesgue measure,

$$\nu(E) = \int_E \frac{1}{x^2}\, dm(x) \text{ and } \mu(E) = \int_E \frac{1}{x}\, dm(x).$$

(a) Prove that $L^2(\nu) \subset L^1(\nu)$. (b) Give an example of some $f(x) \in L^2(\mu) \setminus L^1(\mu)$.

6. (a) Formulate the Dominated Convergence Theorem. (b) Show that the function

$$G(y) = \int_0^\infty 2^{-x} \sin(yx)\, dm(x), \quad y \in \mathbf{R}$$

is continuous, where m is Lebesgue measure.

7. Construct a first category (i.e. meager, a countable union of nowhere dense sets) subset of $\mathbf{R}$ whose complement has Lebesgue measure zero.

8. (a) Formulate some version of the Hahn-Banach theorem for real normed vector spaces. (b) Use it to prove that if $\mathcal{X}$ is a real normed vector space then the bounded linear functionals on $\mathcal{X}$ separate points. (That is, show that if $x, y \in X$ with $x \neq y$ then there exists some $\lambda \in \mathcal{X}^*$ with $\lambda(x) \neq \lambda(y)$.)