PH.D. QUALIFYING EXAM: REAL ANALYSIS

Fall 2025

Solve all seven problems. Show all steps and formulate theorems cited.

1. Suppose that $a \leq a_n < b_n \leq b$, $1 \leq n \leq k$. Suppose also that

$$\#\{n : x \in [a_n, b_n]\} \geq s$$

for every $x \in [a, b]$. Prove that there exists n such that $b_n - a_n \geq \frac{(b-a)s}{k}$.

2. For $n \in \mathbf{N}$ and $x \in [0, 1]$ let $f_n(x) = \frac{2n^2 x}{e^{n^2 x}}$. Find the limit function,

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Prove or disprove that the convergence of f_n to f is uniform on $[0, 1]$.

3. (a) State the Radon-Nikodym theorem.

(b) Let μ be a Borel measure on $[0, 1]$ that is absolutely continuous with respect to Lebesgue measure m, i.e. $\mu << m$. Let $C = \sup\{\mu(E) : m(E) = \frac{1}{2}\}$ and suppose that $C < \infty$. Show that there is a Borel set E with $m(E) = \frac{1}{2}$ and $\mu(E) = C$.

4. Prove that $C([0, 1])$, the space of continuous real valued functions on $[0, 1]$, when equipped with the uniform metric $d(f, g) = \max\{|f - g|\}$, is separable.

5. Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear vector spaces over $\mathbf{R}$, and let $A \subset \mathcal{X}$ with the linear span of A dense in $\mathcal{X}$. Let $T_n : \mathcal{X} \to \mathcal{Y}$ be bounded linear maps such that

$$\sup\{||T_n|| : n \in \mathbf{N}\} = B < \infty.$$

Suppose that $\lim_{n\to\infty} ||T_n a|| = 0$ for every $a \in A$. Prove that $\lim_{n\to\infty} ||T_n x|| = 0$ for every $x \in \mathcal{X}$.

6. Let $C[0, 1]$ be the space of continuous, real valued functions on the unit interval. Prove or disprove that $C[0, 1]$ is a Banach space when equipped with the norm

$$||f|| = \left(\int_0^1 f^2 \, dm\right)^{\frac{1}{2}}.$$

7. Define a measure μ on the set of natural numbers $\mathbf{N}$ by

$$\mu(A) = \sum_{n\in A} \frac{1}{n^2}.$$

(a) Prove or disprove: $L^1(\mu) \subset L^2(\mu)$.
(b) Prove or disprove: $L^2(\mu) \subset L^1(\mu)$.