PH.D. QUALIFYING EXAM: REAL ANALYSIS

January, 2025

Solve five of the seven problems. Show all steps and formulate theorems cited.

1. Let $L^1[0,1]$ and $L^2[0,1]$ be endowed with their respective norm topologies. m is Lebesgue measure. Prove the following:
(a) $\{f \in L^1[0,1] : \int |f|^2 \, dm \leq 1\}$ is closed in $L^1[0,1]$ and has empty interior.
(b) The inclusion map $L^2[0,1] \to L^1[0,1]$ is continuous and not surjective.

2. Let $\{f_n\}$ be a sequence in $L^1(0,\infty)$. Suppose that $f_n(x) \to f(x)$ for a.e. $x \in (0,\infty)$. Is $f(x)$ necessarily integrable? Prove or give a counterexample.

3. Let $a, b \in \mathbf{R}$ with $a < b$, and let $f : [a,b] \to \mathbf{R}$.
(a) What does it mean for f to be absolutely continuous? Give a definition.
(b) Suppose f is absolutely continuous and $f(x) \geq \epsilon > 0$ for every $x \in [a,b]$. Prove that $g = \frac{1}{f}$ is absolutely continuous.

4. Let X and Y be Banach spaces, and suppose that $x_n \to x$ weakly in X. Recall that $\{||x_n||\}$ is necessarily bounded. Let $T : X \to Y$ be a bounded operator.
(a) Prove that $Tx_n \to Tx$ weakly.
(b) Prove that if T is a compact operator then $Tx_n \to Tx$ in the norm topology.

5. Let f be an integrable function defined on a measurable set E in a measure space $(X, \mathcal{A}, \mu)$.
(a) Prove that the set $A = \{x \in E : f(x) \neq 0\}$ is σ-finite.
(b) Suppose that g is another integrable function defined on E, both f and g are real-valued, and that $f^2 + g^2$ is integrable. Prove that fg is integrable.

6. Let m denote Lebesgue measure on $\mathbf{R}$. Recall that for any measurable set $E \subset \mathbf{R}$ with $m(E) > 0$, and any $\epsilon > 0$, there exists a finite interval $I \subset \mathbf{R}$ such that $m(I \cap E) > (1-\epsilon)m(I)$. Use this to prove that if $E, F \subset \mathbf{R}$ with $m(E) > 0$ and $m(F) > 0$ then the set

$$E - F = \{x - y : x \in E,\ y \in F\}$$

contains a nontrivial interval.

7. Prove that a linear functional f on a normed linear space X is bounded if and only if the kernel of f, i.e.

$$\ker f = \{x \in X : f(x) = 0\},$$

is closed.