Math	**Analysis**	**Last Name:** ______________
PhD Qualifying Exam	**2026 Spring**	**First Name:** ______________

- This exam has 10 questions, and some questions have 2 or 3 parts. In case a question has parts, you are supposed to solve every part to get full credit for the problem.
- *Choose any 5 of the 10 questions below to solve. You have the option to submit a solution to more than 5 problems, but you need to specify which 5 we should grade.*
- *Work* each problem on a separate page (but you can work the parts of a problem on the same page). On each page, just indicate the number of the problem, since there is no need to repeat the description of the problem.
- The Lebesgue measure on $\mathbb{R}$ or on a subset of $\mathbb{R}$ is denoted by λ.
- The abbreviated notation $L^p(\mathbb{R})$ refers to the L^p-space with respect to the Lebesgue measure λ.

1. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of $\mathbb{R}$. Assume that for each $n \in \mathbb{N}$, the set U_n is dense in $\mathbb{R}$, that is, for every interval (a, b), with $a < b$, $U_n \cap (a, b) \neq \emptyset$.

   Show that then $\cap_n U_n$ is dense in $\mathbb{R}$.

2. (a) Construct a function f so that $f \in L^2(\mathbb{R})$ but $f \notin L^3(\mathbb{R})$.

   (b) Construct a function f so that $f \in L^3(\mathbb{R})$ but $f \notin L^2(\mathbb{R})$.

3. Let $(V, \|\|)$ be a normed space, and $T : V \longrightarrow V$ be a linear transformation.

   We say the transformation T is *continuous* if, for a sequence $(v_n)_{n\in\mathbb{N}}$ from V, $\lim\limits_{n\to\infty} \|v_n\| = 0$ implies $\lim\limits_{n\to\infty} \|Tv_n\| = 0$.

   We say the transformation T is *bounded* if there is a constant $B > 0$ so that $\|Tv\| \leq B\|v\|$ for every $v \in V$.

   (a) Show that if the linear transformation $T : V \longrightarrow V$ is bounded then it is continuous.

   (b) Show that if the linear transformation $T : V \longrightarrow V$ satisfies $\|Tv\| \leq B\|v\|$ for some constant $B > 0$ and for every $v \in V$ with $\|v\| = 1$ then T is bounded.

   (c) Show that if the linear transformation $T : V \longrightarrow V$ is continuous then it is bounded. (Hint: use an indirect argument)

4. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of orthonormal elements in a complex inner product space $(X, (f, g))$, that is, $\|f_n\| = 1$ for every $n \in \mathbb{N}$ and $(f_n, f_m) = 0$ for $n \neq m$. (Recall that $\|f\| := \sqrt{(f, f)}$.)

   Show that then $\lim\limits_{N\to\infty} \left\| \dfrac{\sum_{n\leq N} f_n}{\sqrt{N \log N}} \right\| = 0$.

5. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\mathbb{R})$ with $\|f_n\|_{L^2} \leq 1$ for every $n \in \mathbb{N}$.

   Show that then $\lim\limits_{n\to\infty} \dfrac{f_n(x)}{n} = 0$ for λ-almost every $x \in \mathbb{R}$.

6. Let (T_n) be a sequence of linear $L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$ operators so that $\|T_n f\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}$ for every $f \in L^1(\mathbb{R})$ and every n. Suppose that there is a dense set $D \subset L^1(\mathbb{R})$ so that $\lim\limits_{n\to\infty} T_n f$ exists in $L^1(\mathbb{R})$-norm for every $f \in D$.

   Show that then $\lim\limits_{n\to\infty} T_n f$ exists in $L^1(\mathbb{R})$-norm for every $f \in L^1(\mathbb{R})$.

Please go on to the next page…

7. Answer true or false. There is no need to show justification, but you are welcome to do so.
   (a) If, for a function $f : \mathbb{R} \to \mathbb{R}$, the set $\{ x \in \mathbb{R} : f(x) > r \}$ is Lebesgue measurable for every rational number r, then f is Lebesgue measurable.
   (b) If, for a function $f : \mathbb{R} \to \mathbb{R}$, the set $\{ x \in \mathbb{R} : f(x) \in S \}$ is Lebesgue measurable for every Lebesgue measurable set S then f is Lebesgue measurable.
   (c) If the $\mathbb{R} \to \mathbb{R}$ function f is Lebesgue measurable then for every Lebesgue measurable set S the set $\{ x \in \mathbb{R} : f(x) \in S \}$ is Lebesgue measurable.
8. Let $S \subset \mathbb{R}$ be a Lebesgue measurable set with $\lambda S > 0$.

   Show that then the difference set $S - S \coloneqq \{ x - y : x, y \in S \}$ contains a non-empty open set.
9. Consider $\mathbb{1}_{\mathbb{Q}}$, that is, $\mathbb{1}_{\mathbb{Q}}$ is the indicator function of the set of rational numbers.

   Show that $\mathbb{1}_{\mathbb{Q}}$ cannot be the pointwise limit of a sequence of continuous $\mathbb{R} \to \mathbb{R}$ functions. (So there is no sequence $(f_n)_{n \in \mathbb{N}}$ of $\mathbb{R} \to \mathbb{R}$ functions, each of which is continuous at every $x \in \mathbb{R}$, and $\lim_n f_n(x) = \mathbb{1}_{\mathbb{Q}}(x)$ for every $x \in \mathbb{R}$.)
10. Let V be a *complex* vectorspace with (complex) inner product (u, v), and let $T : V \to V$ be an isometry, that is, a linear transformation satisfying $(Tv, Tv) = (v, v)$ for every $v \in V$.

    Show that then $(Tu, Tv) = (u, v)$ for every $u, v \in V$.

End of exam.