Statistics Ph.D. Qualifying Exam: Part I

October 27, 2001

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Suppose that X has a uniform distribution on the unit interval $(0, 1)$. Given X, the random variable Y is uniform on the interval $(0, X)$.

   (a) Find $E(X)$, $Var(X)$, $E(Y)$, and $Var(Y)$.

   (b) Find $E(X|Y)$ and $E(Y|X)$.

   (c) Find $Var(X|Y)$ and $Var(Y|X)$.

2. Suppose that X_1, X_2 and X_3 have a trinomial distribution with index n and probability parameters p_1, p_2 and p_3, where $\sum p_j = 1$. The log likelihood function is

$$l(p_1, p_2, p_3) = \sum X_j \log p_j,$$

and the observed values of the X_j's are $32, 46$ and 22.

(a) Find the maximum likelihood estimates of p_j's.

(b) Find the maximum likelihood estimates of p_j's, when the p_j's satisfy the hypothesis

$$p_1 = \theta^2, \quad p_2 = 2\theta(1-\theta), \quad p_3 = (1-\theta)^2.$$

3. Suppose that the joint p.d.f. of two random variables X and Y is $f(x, y) = x+y, \quad 0 \leq x, y \leq 1$.

   (a) Find $P(2X + Y \leq 1)$.

   (b) Find the p.d.f. of $Z = XY$.

4. Let X, Y, U be independent random variables with $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, and $U \sim \text{Uniform}(0,1)$. Let

$$V = \begin{cases} X & , \quad \text{if} \quad U > a, \\ Y & , \quad \text{if} \quad U \leq a, \end{cases}$$

where a is a constant in $(0,1)$. Find

$$P(X + Y = n \mid V = k).$$

5. Let $X_1, \ldots, X_n$ be a random sample of size n from the p.d.f.

$$f(x;\theta) = \theta x^{-2}, \quad 0 < \theta \leq x < \infty.$$

(a) Find the MLE of θ.

(b) Find the method of moment estimator of θ based on $E(X^{1/2})$.

(c) Find the method of moment estimator of θ based on $E(X^{-1})$.

6. Imagine a population of $N+1$ boxes. Box number k contains k red and $N-k$ green balls ($k=0,1,\ldots,N$). A box is chosen at random and n random drawings are made from it, the ball drawn being replaced each time by a ball with the opposite color. Define

Event A: All n balls turn out to be red,
Event B: The $(n+1)$st draw yields a red ball.

(a) Find $P(A|\text{Box } k \text{ is chosen})$ ($k=0,1,\ldots,N$).

(b) Find $P(A)$.

(c) Find $P(A\cap B)$.

(d) Find $P(B|A)$.

(e) Find an approximation to $P(B|A)$, using the fact that if N is large,

$$\frac{1}{N}\sum_{k=1}^{N}(\frac{k}{N})^n \sim \int_0^1 x^n dx = \frac{1}{n+1}.$$

7. Let $X_1, \ldots, X_n$ be i.i.d. from the uniform distribution on the interval $(\theta, \theta + 1)$.

   (a) Find the joint distribution of $X_{(1)}$ and $X_{(n)}$, where $X_{(1)} = \min(X_1, \ldots, X_n)$, and $X_{(n)} = \max(X_1, \ldots, X_n)$.

   (b) Find a UMP test of size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.

8. Let A, B, C be a random sample of size 3 from $U(0,1)$ distribution. Find the probability that

$$Ax^2 + Bx + C = 0$$

has real roots.

9. Let $X_1, \dots, X_n$ be i.i.d. from the density $f(x;\theta) = 3x^2/\theta^3$ if $0 < x < \theta$ and $f(x;\theta) = 0$ otherwise.

   (a) Show that $X_{(n)} = \max(X_1, \dots, X_n)$ is a sufficient and complete statistic for θ.

   (b) Find the UMVUE for θ.

10. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with density $f(x, \theta) = \theta\, x^{\theta-1}, 0 < x < 1, \theta > 0$.

   (a) Derive the UMP (Uniformly Most Powerful) size α test for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

   (b) What is the power function of your test ?

   (c) Show that UMP size-α test for testing $H_0 : \theta = 1$ versus $H_2 : \theta \neq 1$ does not exist.

11. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from Poisson($\lambda\mu$) and Poisson(μ) populations respectively.

(a) Find the MLE's $(\hat{\lambda}, \hat{\mu})$ for (λ, μ).

(b) Find jointly minimal sufficient statistics S for (λ, μ).

(c) Is $(\hat{\lambda}, \hat{\mu})$ a function of S?

(d) Is $(\hat{\lambda}, \hat{\mu})$ jointly sufficient for (λ, μ)?

12. The normally distributed random variables $X_1, \ldots, X_n$ are said to be serially correlated or to follow an autoregressive model if we can write

$$X_i = \theta X_{i-1} + \epsilon_i, \quad i = 1, \ldots, n,$$

where $X_0 = 0$ and $\epsilon_1, \ldots, \epsilon_n$ are independent $N(0, \sigma^2)$ random variables.

(a) Show that the density of $\mathbf{X} = (X_1, \ldots, X_n)'$ is

$$p(\mathbf{x}, \theta) = (2\pi\sigma^2)^{-n/2} \exp\{-(1/2\sigma^2)\sum_{i=1}^{n}(x_i - \theta x_{i-1})^2\},$$

for $-\infty < x_i < \infty$, $i = 1, \ldots, n$ and $x_0 = 0$.

(b) Show that the likelihood ratio statistic of $H_0 : \theta = 0$ (independence) vs $H_1 : \theta \neq 0$ (serial correlation) is equivalent to

$$-(\sum_{i=2}^{n} X_i X_{i-1})^2 / \sum_{i=1}^{n-1} X_i^2.$$