Statistics Ph.D. Qualifying Exam: Part II

November 3, 2001

Student Name: ____________________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal distribution with mean 0 and variance $\sigma^2 > 0$. Define the quadratic forms $s_i = \underset{\sim}{x}' A_i \underset{\sim}{x}$, $i = 1, 2$, where $\underset{\sim}{x} = (X_1, \ldots, X_n)'$ and the A_i 's are symmetric matrices of real numbers.

   (a) Show that if $A_i^2 = A_i$, then s_i/σ^2 is distributed as a central chi-square variable with degrees of freedom $f_i = rank(A_i)$.

   (b) Show that if $A_i A_j = 0$, then s_1 and s_2 are independently distributed of one another.

2. Let X_1 and X_2 be independently and identically distributed from a geometric distribution with probability mass function given by

$$p(x) = P(X = x) = p(1 - p)^{x-1}, \text{ where } x = 1, 2, 3, \ldots.$$

(a) Find the UMVUE for p^2.

(b) Find the UMVUE for $(p + 1)^2/p$.

3. Let $\{X_1, \ldots, X_n\}$ be a random sample from the uniform distribution $U(\theta, \theta + 1)$. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we use the test

$$\text{reject } H_0, \text{ if } \min_{1 \le i \le n} X_i \ge 1 \text{ or } \max_{1 \le i \le n} X_i \ge c,$$

where c is a constant to be determined.

(a) Find c such that the test will have probability of type I error α.

(b) Find the power function.

(c) Is the test UMP test ? Explain.

(d) Find the values of n and c so that it will have level $\alpha = 0.1$ and power at least 0.8 if $\theta > 1$.

4. Let $X_1, X_2, \ldots$ be a sequence of independent Exponential(λ) random variables. Let $N \sim$ Geometric(p) be a geometric random variable that is independent of the X's. Let $X_{(1)}, \ldots, X_{(N)}$ be the order statistics based on a sample of size N.

(a) Prove that

$$P\{X_{(1)} > a\} = \frac{pe^{-\lambda a}}{1-(1-p)e^{-\lambda a}}.$$

(b) Hence, otherwise, find $E(X_{(1)})$.

5. Suppose that diseased trees are distributed randomly and uniformly throughout a large forest with an average of λ per acre. Let $X_1, X_2, \ldots, X_n$ be the numbers of diseased trees observed in n one-acre plots. Let θ be the probability that a random chosen one-acre plot has no tree.

   (a) Find the UMVUE for θ.

   (b) Find the UMVUE for $\lambda\theta$.

6. Let $\{X_1, \ldots, X_m\}$ be a random sample from the normal distribution with mean μ_1 and variance $\sigma^2 > 0$ and $\{Y_1, \ldots, Y_n\}$ a random sample from the normal distribution with mean μ_2 and variance $\sigma^2 > 0$. Assume that the X_i's are independently distributed of the Y_j's.

   (a) Derive the likelihood ratio test procedure for testing $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$.

   (b) What is the sampling distribution of your testing statistics under H_0?

7. Suppose that X_1, X_2, X_3, and X_4 are independently distributed of each other. For $i = 1, 2, 3, 4$, let the probability density function X_i be

$$f_i(x) = \begin{cases} \lambda_i e^{-\lambda_i x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(a) Find $P(\min(X_1, X_2) < \min(X_3, X_4))$ and $P(X_1 < \min(X_2, X_3) < X_4)$.

(b) Now suppose that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/\theta$. Compare the following two estimators of θ: $\hat{\theta_1} = (X_1 + X_2 + X_3 + X_4)/4$ and $\hat{\theta_2} = 4\ \min(X_1, X_2, X_3, X_4)$.

8. Let X follow a Poisson distribution with mean λ and given $X = k$, $Y|X = k$ follows a binomial distribution, $B(k, p)$.

   (a) Find the distribution of Y.

   (b) Show that Y and $X - Y$ are independent.

   (c) Find $P(X = x|Y = y)$.

9. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with density $f(x;\theta), x \in S, \theta \in \Omega$. Denote by $L = L(\theta|\underset{\sim}{X}) = \prod_{i=1}^{n} f(X_i;\theta)$. Suppose that the following conditions hold:

   (i) $s = \frac{1}{L}\frac{\partial}{\partial\theta}L$ exists for all $X_i \in S$ and all $\theta \in \Omega$.

   (ii) $\lambda = E(s^2|\theta) > 0$ for all $\theta \in \Omega$.

   (iii) It is permissible to interchange the operator E and $\frac{\partial}{\partial\theta}$. That is, $\frac{\partial}{\partial\theta}E = E\frac{\partial}{\partial\theta}$.

   (a) Show that the statistic u is the UMVUE of θ if and only if the following condition holds:
   $$\frac{1}{L}\frac{\partial}{\partial\theta}L(\theta|\underset{\sim}{X}) = C(\theta)(u - \theta),$$
   where $C(\theta)$ is a function of θ but independent of $X_i, i = 1, \ldots, n$.

   (b) Given that the condition in (a) holds, obtain the variance of u.

10. Let $X_1, X_2, ...X_n$ be a random sample from a population with probability function

$$f(x;\theta) = \theta(1-\theta)^x, x = 0, 1, 2, 3, ...$$

where θ is an unknown parameter in $(0, 1)$, under the classical approach. One can also use a Bayesian approach by assuming θ has a prior distribution $\theta \sim U(0, 1)$.

(a) Compute the Cramer-Rao Lower Bound for unbiased estimators of θ.

(b) Find the UMVUE for θ, if possible.

(c) If the loss function $L(\theta, a) = (\theta - a)^2$, find the Bayes estimator of θ.

(d) If the loss function $L(\theta, a) = \frac{(\theta-a)^2}{\theta(1-\theta)}$, find the Bayes estimator of θ.

11. Let $Y_1, \dots, Y_N$ be independent random variable such that $Y_i \sim \text{Binomial}(n_i, p_i)$, where

$$p_i = \frac{1}{1 + e^{\alpha + \beta x_i}},$$

and x_i is a fixed covariate, $i = 1, \dots, N$.

(a) Find a set of jointly sufficient statistics for (α, β).

(b) Suppose $\tilde{\alpha}, \tilde{\beta}$ are the estimates of α and β that minimize

$$Q = \sum_{i=1}^{N} n_i \hat{p}_i (1 - \hat{p}_i)(\alpha + \beta x_i - l_i)^2,$$

where $\hat{p}_i = Y_i / n_i$ and l_i is some function of $Y_i / n_i, i = 1, \dots, N$. Find $\tilde{\alpha}$ and $\tilde{\beta}$.

(c) Let $\hat{\alpha}, \hat{\beta}$ be MLE's of α and β. Give an argument to show that

$$\text{Mean Square Error}(\hat{\alpha} + \hat{\beta}) \leq \text{Mean Square Error}(\tilde{\alpha} + \tilde{\beta})$$

12. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from Exponential(λ) and Exponential(μ) populations respectively.

   (a) Construct a likelihood ratio test of

$$H_0 : \lambda = \mu \quad \text{versus} \quad H_1 : \lambda \neq \mu.$$

   (b) Give the critical values of this test in terms of percentiles of one of the standard distributions.