Statistics Ph.D. Qualifying Exam: Part I

November 2, 2002

Student Name: ____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal population with mean μ and variance σ^2. Assume that **a priori** μ and σ^2 are independently distributed of each other. Let the prior density of σ^2 be given by

$$P\{\sigma^2\} \propto (\sigma^2)^{-3} e^{-5/\sigma^2}, \sigma^2 > 0.$$

   (a) Assuming non-informative uniform prior for μ, derive the posterior density of σ^2.

   (b) Derive the $(1-\alpha)$ % HPD (Highest Posterior Density) Bayesian interval for σ^2.

   (c) Let the loss function of the estimator $\hat{\sigma}^2$ of σ^2 be given by $l(\sigma^2, \hat{\sigma}^2) = \frac{(\sigma^2 - \hat{\sigma}^2)^2}{\sigma^2}$. Derive the Bayese Estimator of σ^2.

2. Let $\{X_1, \ldots, X_m\}$ be a random sample from the normal population with mean μ_1 and variance σ_1^2. Let $\{Y_1, \ldots, Y_n\}$ be a random sample from the normal population with mean μ_2 and variance σ_2^2 independently of $\{X_1, \ldots, X_m\}$.

   (a) Derive the size α Likelihood Ratio test for testing $H_0 : \sigma_1^2 = \sigma_2^2$ vs $H_1 : \sigma_1^2 \neq \sigma_2^2$.

   (b) Derive the power function of your test.

   (c) Derive a $1 - \alpha$ % confidence interval for $\theta = \sigma_1^2/\sigma_2^2$. If you use this confidence interval to test the above hypothesis H_0, how is this compared with the procedure of (a)?

3. Let $X_1, X_2, \ldots,$ be a sequence of independent identically distributed exponential random variables with parameter λ. Let N be a geometric random variable with parameter p. Assume that the X's and N are independent. Find the following

   (a) $E(\frac{1}{N}\sum_{i=1}^{N} X_i)$

   (b) $P(X_{(N)} > a)$

   (c) $E(X_{(N)})$

4. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from Poisson$(\lambda\mu)$ and Poisson(μ) populations respectively.

   (a) Find the MLE's $(\hat{\lambda}, \hat{\mu})$ for (λ, μ).

   (b) Find jointly sufficient statistics S for (λ, μ).

   (c) Is $(\hat{\lambda}, \hat{\mu})$ a function of S?

   (d) Is $(\hat{\lambda}, \hat{\mu})$ jointly sufficient for (λ, μ)?

5. Suppose that the primary endpoint of a clinical trial is survival time T and the distribution of T is negative exponential with p.d.f. $f(t;\theta) = \frac{1}{\theta}e^{-\frac{t}{\theta}}, t > 0$. Suppose further that the study primary objective is formulated into testing $H_0 : \theta = 3$ vs $H_1 : \theta = 1$. Determine the sample size for the clinical trial to obtain a level of 5% test with power 80%, by using

   (a) the exact method, and

   (b) the normal approximation.

6. The normally distributed random variables $X_1, \ldots, X_n$ are said to be serially correlated or to follow an autoregressive model if we can write

$$X_i = \theta X_{i-1} + \epsilon_i, \quad i = 1, \ldots, n,$$

where $X_0 = 0$ and $\epsilon_1, \ldots, \epsilon_n$ are independent $N(0, \sigma^2)$ random variables. Show that the density of $\mathbf{X} = (X_1, \ldots, X_n)'$ is

$$p(\mathbf{x}, \theta) = (2\pi\sigma^2)^{-n/2} \exp\{-(1/2\sigma^2)\sum_{i=1}^{n}(x_i - \theta x_{i-1})^2\},$$

for $-\infty < x_i < \infty$, $i = 1, \ldots, n$ and $x_0 = 0$.

7. Suppose $X \sim f(x;\theta)$, where θ takes $0, 1/2, 1, 3/2$ or 2. The following table gives the evaluations of the likelihood function of θ given a set of i.i.d. observations.

θ	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$L(\theta)$	0.079	0.313	0.606	0.372	0.054

(a) Find the maximum likelihood estimate of θ.

(b) Conduct a level of 5% likelihood ratio test on $H_0 : \theta = 0$ vs H_1 : otherwise. ($\chi^2_{0.05}(1) = 3.84$ and $\chi^2_{0.05}(2) = 5.99$)

8. Let $X_1, \ldots, X_n$ be i.i.d. from the following probability mass function

$$p(x) = P(X = x) = \frac{n!}{x!(n-x)!} \frac{\theta^x}{(1+\theta)^n}, \quad \text{where } x = 0, 1, \ldots, n.$$

Let $\phi = (1+\theta^n)/(1+\theta)^n$ be the parameter of interest.

(a) Find an unbiased estimator for ϕ.

(b) For the case $n = 2$, find the UMVUE for ϕ.

9. Let $X_1, \ldots, X_n$ be i.i.d. from the uniform distribution on the interval $(\theta - 1, \theta + 1)$.

(a) Find the joint distribution of $X_{(1)}$ and $X_{(n)}$.

(b) Find a UMP test of size α for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$.

10. Let $X_1, \ldots, X_n$ be a random sample of size n from a population with density

$$f(x;\theta,\mu) = \frac{1}{3\theta}e^{-x/\theta} + \frac{2}{3\mu}e^{-x/\mu}, \quad x > 0, \theta > 0, \mu > 0.$$

(a) Use Method of Moments to find estimators of θ and μ.

(b) Explain how to find the asymptotic distributions of the estimators found above.

11. Let X_1 and X_2 be two independent random variables with the same density

$$f(x) = xe^{-x}, \quad x > 0.$$

Compute

(a) $E\left(\frac{X_1}{X_2} | X_1 < X_2\right)$.

(b) $E\left(\frac{\min(X_1,X_2)}{\max(X_1,X_2)}\right)$

12. Let $X_1, \ldots, X_n$ be a random sample of size n from a population with density

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \quad x > 0, \theta > 0.$$

We wish to estimate $\tau = P(X_1 > 1) = e^{-1/\theta}$.

(a) Compute the Cramer-Rao lower bound for the variance of unbiased estimator of τ.

(b) Find the maximum likelihood estimator of τ.

(c) Find the uniformly minimum variance unbiased (UMVU) estimator of τ.