Statistics Ph.D. Qualifying Exam: Part II

October 25, 2003

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let $\{X_1, \ldots, X_n\}$ be a random sample from a population with density $f(x, \theta) = \{\theta^2 + 2\theta\,(1-\theta)\}^x (1-\theta)^{2(1-x)}, x = 0, 1, 0 < \theta < 1$.

   (a) Derive the moment estimator of θ.

   (b) Derive the MLE of θ by using the EM-algorithm.

   (c) Which estimator you would prefer for estimating θ?

2. Let $X_1, \ldots, X_n$ be a random sample from a Normal $(\mu, 1/\tau)$ population. Assume the following prior specifications on μ and τ: $\mu|\tau \sim N(\mu_0, \frac{1}{\lambda_0 \tau})$, $\tau \sim Gamma(\alpha_0, \beta_0)$, with p.d.f.

$$\frac{1}{\Gamma(\alpha_0)\beta_0^{\alpha_0}} \tau^{\alpha_0 - 1} e^{-\tau/\beta_0},$$

where $\mu_0, \alpha_0, \beta_0, \lambda_0$ are constants.

(a) Show that this prior specification is conjugate for this problem.

(b) Find the posterior distribution of μ.

3. Let Y_i denote the response of a subject at time i, $i = 1, \ldots, n$. Suppose that Y_i satisfies the following model

$$Y_i = \theta + \epsilon_i, \quad i = 1, \ldots, n,$$

where ϵ_i can be written as $\epsilon_i = ce_{i-1} + e_i$ for given constant c, $0 \le c \le 1$, and the e_i are independent identically distributed with mean zero and variance σ^2, $i = 1, \ldots, n$; $e_0 = 0$ (the ϵ_i are called moving average errors). Let

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i, \quad \hat{\theta} = \sum_{i=1}^{n} a_j Y_j,$$

where

$$a_j = \sum_{i=0}^{n-j}(-c)^i\left(\frac{1-(-c)^{j+1}}{1+c}\right)\Big/\sum_{i=1}^{n}\left(\frac{1-(-c)^i}{1+c}\right)^2.$$

(a) Show that $\hat{\theta}$ is the weighted least squares estimate of θ.

(b) Show that if $e_i \sim N(0, \sigma^2)$, then $\hat{\theta}$ is the MLE of θ.

(c) Show that $\bar{Y}$ and $\hat{\theta}$ are both unbiased but $Var(\hat{\theta}) \le Var(\bar{Y})$, while the inequality holds unless $c = 0$.

4. Let $X_1, \ldots, X_n$ be i.i.d. having the uniform distribution on the interval $(\theta_1 - \theta_2, \theta_1 + \theta_2)$. Here we assume that θ_2 is positive.

   (a) Find the UMVUE's for θ_1 and θ_2.

   (b) Find the UMVUE for θ_1/θ_2.

5. Let $\{X_1, \ldots, X_n\}$ be a random sample from a continuous probability distribution with density $f(x;\theta)$.

   (a) Consider the problem of testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Show that the size α MP (Most Powerful) test for testing H_0 vs H_1 is given by:
   Reject H_0 at level α if $\frac{L_1}{L_0} \geq k$ for some constant $k > 1$, where $L_i = \prod_{j=1}^{n} f(X_j;\theta_i), i = 0, 1$.

   (b) Given that UMP (Uniformly Most Powerful) tests for testing $H_0 : \theta = \theta_0$ vs $H_2 : \theta > \theta_0$ exist, illustrate how one would use result of (a) to derive the size α UMP test for testing H_0 vs H_2.

6. For the linear model

$$Y = X\beta + \epsilon,$$

such that $E(\epsilon) = 0$ and $Cov(\epsilon) = \sigma^2 I$

(a) State the Gauss-Markov theorem.

(b) Prove Gauss-Markov Theorem.

7. Assume that random variables X and Y follow a trinomial distribution with the probability distribution function

$$P(X = x, Y = y|\theta, \lambda) = \frac{n!}{x!y!(n-x-y)!}\theta^x \lambda^y (1-\theta-\lambda)^{n-x-y}, \quad x \geq 0, y \geq 0, x+y \leq n.$$

Suppose further that (θ, λ) have a uniform prior distribtuion over the triangular region $0 < \theta + \lambda < 1$.

(a) Find the posterior distribution of (θ, λ) given X and Y.

(b) Find the Bayes estimator of θ under square error loss function.

(c) Find the Bayes estimator of $\theta + \lambda$ under square error loss function.

8. If $X_1, \ldots, X_n$ is a random sample of size n from an exponential population

$$
\begin{aligned}
f(x;\theta,\sigma) &= (1/\sigma)e^{-(x-\theta)/\sigma} \quad , \quad \theta \le x < \infty, \\
&= 0 \text{ otherwise.}
\end{aligned}
$$

Let $\bar{X}$ be the sample mean and $X_{(1)}$ be the smallest order statistic.

(a) Find the joint p.d.f. of $\bar{X} - X_{(1)}$ and $X_{(1)}$.

(b) Find the marginal p.d.f.s of $\bar{X} - X_{(1)}$ and $X_{(1)}$.

(c) Are $\bar{X} - X_{(1)}$ and $X_{(1)}$ independently distributed ?

(d) Find the UMVUE of θ.

9. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}.$$

(a) Find the M.L.E. of $e^{-t/\theta}$, where t is a positive constant.

(b) Find the UMVU estimator of $e^{-t/\theta}$.

(c) Is the UMVU estimator of $e^{-t/\theta}$ an efficient estimator ?

10. If a random vector (U, V) is uniformly distributed over the region

$$C = \left\{ (u, v) | 0 \le u \le \sqrt{h\left(\frac{v}{u}\right)} \right\},$$

where $h(y)$ is a nonnegative function.

(a) Show that the marginal p.d.f. of $Y = V/U$ is proportional to h.

(b) Explain how to generate a Cauchy distribution using the above result with $h(y) = \frac{1}{1+y^2}$.

(c) Describe another method to generate a Cauchy distribution.

11. Let $X_1, \ldots, X_m$ be a random sample from $N(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_n$ a random sample from $N(\mu_2, 4\sigma^2)$.

(a) Obtain maximum likelihood estimators of μ_1, μ_2, and σ^2.

(b) Derive the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ vs. $H_1 : \mu_1 \neq \mu_2$. What is the sampling distribution of your test statistic under H_0?

(c) Obtain a $100(1 - \alpha)\%$ confidence interval for σ^2.

12. Let $X_1, X_2, \ldots, X_m$ be i.i.d. uniformly distributed on the interval $(0, \theta)$, and $Y_1, \ldots, Y_n$ i.i.d. uniform on $(0, \tau)$. Assume the X's and Y's are mutually independent.

(a) Derive the density for

$$U = (\max X_i)/(\max Y_i).$$

(b) Suppose we wish to test the hypotheses

$$H_0 : \theta \leq \tau \text{ vs. } H_1 : \theta > \tau.$$

Consider the test which rejects when $U > c$ for some given critical value c. Show that the power of the test is monotonically increasing in $\rho = \theta/\tau$.

(c) If $m = 4$ and $n = 2$, find the critical value of the test in (b) so as to achieve level $\alpha = 1/24$.