Statistics Ph.D. Qualifying Exam: Part I

November 13, 2004

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

1	2	3	4	5	6	7	8	9	10	11	12

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be exponential random variables with X's having mean $1/(\theta\lambda)$ and Y's having mean $1/\lambda$. Let $X = \sum_{i=1}^{n} X_i$ and $Y = \sum_{i=1}^{m} Y_i$. Let $T = X/Y$ and $U = Y$.

   (a) Find the joint density of T and U.

   (b) Find the marginal density of T.

   (c) Now let T_1, T_2 be a random sample with the same distribution as T. Find the maximum likelihood estimate of θ.

2. Let $Y_1, \ldots, Y_n$ be independent random variables such that

$$Y_i = \mu + \sum_{j=1}^{p} x_{ij}\beta_j + \epsilon_i,$$

$i = 1, \ldots, n$, and $p < n$. Let $\mathbf{X}$ be the $n \times p$ matrix of x_{ij}'s, $\mathbf{1}$, the $n \times 1$ vector of 1's, β the $p \times 1$ vector of β_j's and ϵ, the vector of ϵ_i's. Assume that $\mathbf{X}^T\mathbf{1} = \mathbf{0}$.

(a) Derive an expression for $\hat{\beta}$, the least squares estimate of β.

(b) Describe the test of

$$H_0 : \beta = \mathbf{0}$$

(c) How would you check the assumption that $\epsilon \sim N(\mathbf{0}, \sigma^2\mathbf{I})$?

3. Let X_1 and X_2 be independent Poisson random variables with means $\lambda\mu$ and μ respectively. We wish to test

$$H_0 : \lambda \leq \lambda_0 \text{ versus } H_1 : \lambda > \lambda_0,$$

where μ is unknown.

(a) Explain why the appropriate test is a conditional test, based on the conditional distribution of X_1 given $X_1 + X_2$, and derive the test procedure.

(b) If $X_1 = x_1$ and $X_2 = x_2$ are observed, and $s = x_1 + x_2$, show that the P-value corresponding for this test is approximately

$$1 - \Phi\left[\frac{x_1 - s\lambda_0/(1+\lambda_0)}{\{s\lambda_0/(1+\lambda_0)^2\}^{1/2}}\right]$$

4. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal distribution with mean μ and variance σ^2. Put $U_1 = \sum_{i=1}^n X_i$, $U_i = X_1 - X_i$, $i = 2, \ldots, n$ and $S^2 = \sum_{i=1}^n (X_i - U_1/n)^2$.

   (a) Show that $\{U_2, \ldots, U_n\}$ is independently distributed of U_1 stochastically. What is the joint probability distribution of $\{U_i, i = 2, \ldots, n\}$?

   (b) Express S^2 as a function of $\{U_i, i = 2, \ldots, n\}$ and then derive the probability density distribution of S^2.

5. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal distribution with mean μ_1 and variance σ_x^2 and $\{Y_1, \ldots, Y_m\}$ a random sample from the normal distribution with mean μ_2 and variance σ_y^2. Assume that the X_i's are independently distributed of the Y_j's. Put $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$, $\bar{Y} = \frac{1}{m}\sum_{i=1}^{m} Y_i$, $S_y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y})^2$, $\hat{\sigma}^2 = \frac{1}{n}S_x^2 + \frac{1}{m}S_y^2$.

   (a) What are a and b if one approximates the sampling distribution of $\hat{\sigma}^2$ by $a\chi_b^2$, where χ_b^2 is a central Chi-square random variable with degrees of freedom b.

   (b) Derive a $(1 - \alpha)$ % approximate confidence interval for $\mu_1 - \mu_2$ by using the approximation in (a).

6. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta)$ with parameter space $\Omega = (a, b), a < 0 < b$. Let $l(\hat{\theta}, \theta)$ be the loss function of the estimator $\hat{\theta}$ of the parameter θ.

   (a) Show that if $l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$, then the Bayese estimator of θ is given by the posterior mean of θ.

   (b) If $l(\hat{\theta}, \theta) = |\hat{\theta} - \theta|$, what is the Bayese estimator of θ?.

7. Let $\{X_{i,1}, \ldots, X_{i,n_i}\}$ be independent random samples from the probability distribution with density $f(x, \theta_i, \phi_i) = \frac{1}{\phi_i} e^{-\frac{1}{\phi_i}(x-\theta_i)}, x \geq \theta_i,$ where $\phi_i > 0, i = 1, 2$, respectively.

   (a) Derive the level-α likelihood ratio test for testing $H_0 : \phi_1 = \phi_2$ vesus $H_1 : \phi_1 \neq \phi_2$ when $\{\theta_i, i = 1, 2\}$ are unknown.

   (b) Derive the sampling distribution of your testing statistic under H_0.

   (c) Derive the power function of your test.

8. Suppose that $X_1, X_2, ...X_n$ form a random sample from a uniform distribution with the following p.d.f.

$$f(x;\theta) = \frac{1}{\theta}, \qquad 0 \le x \le \theta.$$

Let $Y_n = \max_{1 \le i \le n} X_i$ and we would like to test $H_0 : 2 \le \theta \le 4$ vs. $H_1 : \theta < 2$ or $\theta > 4$. The critical region is chosen to be $C = \{Y_n < 3 \text{ or } Y_n > 4\}$.

(a) Determine the power function of the test.

(b) Determine the size (maximum type I error) of the test.

9. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with the following p.d.f.

$$f(x;\theta) = \theta(1+x)^{-(1+\theta)}, \quad x > 0, \theta > 0.$$

(a) Estimate θ by the method of moments assuming $\theta > 1$.

(b) Find the maximum likelihood estimator of θ.

10. Let $X_1, X_2, \cdots, X_n$ be a random sample from a geometric distribution with p.d.f.

$$f(x;\theta) = \theta(1-\theta)^x, \quad x = 0, 1, 2, \cdots, 0 < \theta < 1.$$

(a) Find the Cramer-Rao lower bound for the variance of unbiased estimators of $(1-\theta)/\theta$.

(b) Find the UMVUE of $(1-\theta)/\theta$, if such exists.

(c) Find the UMVUE of θ, if such exists.

11. Let X, Y and Z be three independent random variables, each uniformly distributed over $(0, 1)$.

    (a) Find the distribution of XY.

    (b) Find the distribution of $\frac{XY}{Z}$.

12. Let $X_1, X_2, \cdots, X_n$ be a random sample from $Ber(1, \theta)$ Bernoulli distribution for which the θ is unknown $(0 < \theta < 1)$.

   (a) Find the MLE of θ^2.

   (b) Find the UMVUE of θ^2.

   (c) Compare the asymptotic distributions of MLE and UMVUE found above.