Statistics Ph.D. Qualifying Exam: Part I

November 12, 2005

Student Name: ____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary.

3. Assemble your work in right order and in the original problem order.

1. Let X and Y be two independent random variables following Beta(a, b) and Beta($a+b$, c) distributions, respectively. Consider the transformation $U = XY$ and $V = X$.

   (a) Find the joint p.d.f. of U and V.

   (b) Find the marginal p.d.f. of U.

2. Suppose that the conditional distribution of Y given θ is a Poisson distribution of mean θ and the distribution of θ is an exponential distribution with mean 1.

   (a) Find the mean and variance of Y.

   (b) Find the marginal distribution of Y.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from $\mathrm{N}(\theta,\ c\theta^2)$, where c is a known constant.

   (a) Find a minimal sufficient statistics for θ.

   (b) Is the above minimal sufficient statistics for θ complete ? You need to justify your answer.

4. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a $U(0,1)$ distribution.

(a) Find the p.d.f. of $Y_i = -\ln(X_i)$.

(b) Find the p.d.f. of $Y = \sum_{i=1}^{n}[-\ln(X_i)]$.

(c) Find the p.d.f. of $\prod_{i=1}^{n} X_i$.

5. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from Exponential(λ) and Exponential(μ) populations with mean λ and μ, respectively.

   (a) Derive a likelihood ratio test for testing $H_0 : \lambda = \mu$ versus $H_1 : \lambda \neq \mu$,

   (b) Give a critical value of the test in terms of some commonly used/well-known statistics table.

6. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal distribution with mean μ and variance σ^2. Put $Q_i = \underset{\sim}{X}' A_i \underset{\sim}{X}$, $i = 1, 2$, where $\underset{\sim}{X}' = (X_1, \ldots, X_n)$ and the A_i are symmetric matrices of real numbers.

(a) Show that if $A_1 A_2 = 0$, then Q_1 and Q_2 are distributed independently of each other stochastically.

(b) Show that if $A_i^2 = A_i$ and if $A_i \underset{\sim}{1} = \underset{\sim}{0}$, where $\underset{\sim}{1}$ is a column vector of 1's and $\underset{\sim}{0}$ a column vector of 0's, then Q_i is distributed as $\sigma^2 \chi^2_{r_i}$, where $r_i = Rank A_i$ and $\chi^2_{r_i}$ is a central chi-square random variable with degrees of freedom r_i.

7. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal distribution with mean μ_1 and variance $4\sigma^2$ and $\{Y_1, \ldots, Y_m\}$ a random sample from the normal distribution with mean μ_2 and variance $9\sigma^2$, where σ^2 is unknown.

   (a) Derive a $(1-\alpha)\%$ confidence interval for $\delta = \mu_1 - \mu_2$.

   (b) Assuming Jeffrey's non-informative prior $P(\mu_1, \mu_2, \sigma^2) \propto \sigma^{-2}$, derive a $(1-\alpha)$ % HPD (Highest Posterior Density) Bayesian interval for $\delta = \mu_1 - \mu_2$. How is this HPD interval comparing with the confidence interval obtained in (a) above?

   (c) Illustrate how you will use the above results to test the hypothesis $H_0 : \mu_1 = \mu_2$.

8. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \theta(1-\theta)^x, x = 0, 1, \ldots, \infty; 0 < \theta < 1$.

   (a) Obtain a sufficient and complete statistic for θ.

   (b) Obtain the UMVUE (Uniformly Minimum Varianced Unbiased estimator) of θ. What is the UMVUE for $\phi = 1/\theta$?

   (c) Let the prior distribution of θ be given by:

   $$p(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}, 0 < \theta < 1, a > 0, b > 0.$$

   derive the Bayese estimator of θ and the Bayese estimator of $\phi = 1/\theta$ by assuming squared loss function.

   (d) If the loss function of $\hat{\theta}$ is $l(\hat{\theta}, \theta) = |\hat{\theta} - \theta|$, what is the Bayese estimator of θ?.

9. Let $\{X_1, \ldots, X_n\}$ be a random sample from the probability distribution with density $f(x, \theta, \phi) = \phi(x - \theta)^{\phi - 1}, \theta < x < 1 + \theta, \phi > 0$. Assume that θ is known.

   (a) Derive the level-α UMP (Uniformly Most Powerful) test for testing $H_0 : \phi = 1$ versus $H_1 : \phi > 1$.

   (b) Derive the sampling distribution of your testing statistic under H_0.

   (c) Derive the sampling distribution of your testing statistic under H_1 and hence the power function of your test.

10. Let $X_1, \ldots, X_n$ be a random sample from a $\mathcal{U}(\lambda, \theta)$.

(a) Find the jointly sufficient statistics for the (λ, θ)

(b) Find the MLE's of (λ, θ) and show that they are jointly complete and sufficient.

(c) Find the best unbiased estimator of $(\lambda + \theta)/2$.

11. Let $X_1, \ldots, X_n$ be a random sample from $\mathcal{P}\text{oisson}(\theta)$.

(a) Find an unbiased estimator $d(\mathbf{X})$ of $\theta e^{-2\theta}$.

(b) Find the Cramer-Rao lower bound for all unbiased estimator of $\theta e^{-2\theta}$, and show that $d(\mathbf{X})$ does not attain the Cramer-Rao lower bound.

(c) Find the UMVUE estimator of $\theta e^{-2\theta}$. Does this estimator attain the Cramer-Rao lower bound?

12. Suppose that X is a sample of size one from a $\mathcal{B}\text{eta}(1, \theta)$ population, $\theta > 0$.

(a) For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, find the size and sketch the power function of a test procedure which rejects H_0 when $X > 0.75$.

(b) Now take a random sample of size $n : X_1, \ldots, X_n$. Is there a UMP test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$? If so, find it explicitly. If not, prove that it does not exist.