Statistics Ph.D. Qualifying Exam: Part I

January 5, 2008

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

1. Let $f(x, y) = c, x^2 \leq y \leq 1, 0 \leq x \leq 1$, be the joint p.d.f. of X and Y, where c is a constant to be determined.

   (a) Find c.

   (b) Find $P(X \leq Y)$.

   (c) Find the p.d.f. of $Z = X^2$.

2. Let μ_X and σ_X be the mean and standard deviation of a random variable X.

   (a) State and prove Chebyshev's inequality.

   (b) Compare the bound from Chebyshev's inequality for $k = 2$ by calculating

   $$P(|X - \mu_X| \geq k\sigma_X)$$

   for $X \sim U(0, 1)$, and $X \sim Exp(1)$, an exponential distribution with mean 1.

3. Let X, Y be two random variables with a joint pdf

$$f(x,y) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} x^{a-1} y^{b-1} (1-x-y)^{c-1},$$

where $0 < x < 1;\quad 0 < y < 1;\quad 0 < x+y < 1$, a, b, c are positive constants and $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

(a) Derive the marginal distributions of X and Y.

(b) Derive the distribution of $X + Y$.

(c) Derive the conditional distributions of Y given $X = x$.

4. Let $X_1, X_2, \ldots, X_n$ be iid Poisson random variables with unknown mean λ. Let $\theta = P(X_1 = 1)$.

   (a) Find a uniformly minimum variance unbiased estimator T_n of θ.

   (b) Find the asymptotic distribution of T_n.

5. Let $X_1, \ldots, X_m$ be a random sample from $N(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_n$ a random sample from $N(\mu_2, c^2\sigma^2)$ where c^2 is a known positive number.

   (a) Obtain maximum likelihood estimators of μ_1, μ_2, and σ^2.

   (b) Derive the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ vs. $H_1 : \mu_1 \neq \mu_2$.

6. Let $\{X_1, \ldots, X_n\}$ be a random sample from the normal distribution with mean 0 and variance σ^2. Put $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2$.

   (a) Derive the joint probability distribution of $\bar{X}$ and S^2.

   (b) What is the probability distribution of $F = \bar{X}^2/S^2$?

7. Let $X_1, \ldots, X_m$ be a random sample from an **Uniform** distribution over $(0, \theta_1)$, $\theta_1 > 0$ and $Y_1, \ldots, Y_n$ a random sample from an **Uniform** distribution over $(0, \theta_2)$, $\theta_2 > 0$. Assume that the $X_i's$ are independently distributed of the $Y_j's$.

   (a) Obtain a mininmun set of sufficient and complete statistics for $\{\theta_i, i = 1, 2\}$.

   (b) Find the maximum likelihood estimator of $\phi = \theta_1 - \theta_2$.

   (c) Obtain the UMVUE estimator of ϕ.

8. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independently and idnetically distributed as (X, Y), where (X, Y) follows a bivariate normal distribution with means $EX = \mu_1$ and $EY = \mu_2$ and with variances and covariance as $Var(X) = \sigma_1^2$, $Var(Y) = \sigma_2^2$ and $Cov(X, Y) = \rho\sigma_1\sigma_2$, respectively. Put $Z_i = X_i - Y_i, i = 1, \ldots, n$.

   (a) Based on the observed Z_i values, derive the $(1 - \alpha)$ % confidence interval for $\theta = \mu_1 - \mu_2$ in terms of the central-t distribution.

   (b) Illustrate how you would use the result in (a) to derive a test procedure for testing the null hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_1 : \mu_1 \neq \mu_2$.

9. Let $X_1 \ldots, X_{n+1}$ be a random sample from a population with density

$$f(x|\theta) = e^{-(x-\theta)}, \quad x \geq \theta,$$

Assume that the prior density for θ is exponential with mean 1.

(a) Find the posterior density of θ.

(b) Using squared error loss function, find the Bayes estimator of θ.

(c) Compare the Bayes estimator of θ with the maximum likelihood estimator of θ as n increases.

(d) What is the limit of the Bayes estimator as $n \to \infty$.

10. Let N be a nonnegative integer valued random variable.

(a) Prove that

$$E(N) = \sum_{k=0}^{\infty} P(N > k).$$

(b) Suppose that you perform independent Bernoulli trials $\{X_n, n \geq 0\}$ such that $P(X_n = 1) = 1 - \frac{1}{n}$, with

$$X_n = 1, \text{ if success, and } X_n = 0, \text{ if failure;}$$

(thus, the probability of success is not fixed but increases with each trial).

Let N be the number of trials needed to get the first success. Show that $E(N) = e = 2.718...$

11. Let $X_1, \ldots, X_n$ be a random sample from Poisson(λ). Let $d_1(\mathbf{X}) = \sum_{i=1}^n X_i/n$ and $d_2(\mathbf{X}) = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$.

(a) Is either of these two estimators an unbiased estimator of λ? (Fully justify your answer.)

(b) Is either of these two estimators UMVUE of λ? (Fully justify your answer.)

(c) Does either of these two estimators achieve the Cramer-Rao Lower bound? (Fully justify your answer.)

(d) If the answer to the above question is no, which estimator is better and why?

12. Let $Y_1, \ldots, Y_n$ be a random sample from $N(\mu, \sigma^2)$. Consider the problem of testing $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$.

   (a) Show that the likelihood ratio statistic can be expressed in the form

$$\Lambda(T) = n[-\log(1 + T) + T],$$

   where T is a statistics and $nT + n \sim \chi^2(n - 1)$, under H_0.

   (b) Let $W = nT + n, C_1 = \chi^2_{\frac{\alpha}{2}}(n - 1)$ and $C_2 = \chi^2_{1-\frac{\alpha}{2}}(n - 1)$. Show that the test which rejects H_0 if $W > C_1$ or $W < C_2$ has size α and power greater than α at any $\sigma^2, \sigma^2 \neq \sigma_0^2$.