Statistics Ph.D. Qualifying Exam: Part I

November 13, 2010

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. If (X_1, X_2) is distributed as bivariate multinomial with parameters (n, p_1, p_2). That is,

$$\Pr(X_1 = x_1, X_2 = x_2) = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3},$$

where x_i's are non-negative integers satisfying $x_1 + x_2 + x_3 = n$, and $p_3 = 1 - p_1 - p_2$. Derive the following

   (a) $E(X_1)$ and $E(X_2)$.

   (b) $Var(X_1)$ and $Var(X_2)$.

   (c) $Cov(X_1, X_2)$.

2. Let X be a random variable with the p.d.f.

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Find the c.d.f. of X.

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(c) Estimate the sample size n needed in order for the probability to be at least 0.90 that the sample average will be between 0.8 and 1.2.

3. Let B and C be two iid random variables with $U(-1, 1)$ distribution. Find the probability that $x^2 + Bx + C = 0$ has two real roots.

4. Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with probability function

$$f(x|\theta) = \theta(1-\theta)^x, x = 0, 1, 2, ...$$

where $\theta \in \Omega = (0, 1)$.

(a) Suppose θ has a prior distribution $\theta \sim \text{beta}(\alpha_0, \beta_0)$, where α_0, β_0 are known constants. If the loss function $L(\theta, a) = \frac{(\theta-a)^2}{\theta(1-\theta)}$, find the Bayes estimator of θ.

(b) Find the Cramer-Rao lower bound for unbiased estimators of θ.

5. Let X_1, X_2, X_3 be i.i.d. random variables with p.d.f. $f(x) = e^{-x} \quad , x > 0$. Let

$$Y_1 = \frac{X_1}{X_1 + X_2}$$

$$Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3},$$

and

$$Y_3 = X_1 + X_2 + X_3.$$

(a) Find the marginal p.d.f. of Y_1, Y_2 and Y_3.

(b) Are Y_1, Y_2 and Y_3 independent ? Justify your answer.

6. Let $X_1, X_2, ..., X_n$ be i.i.d. random variables with p.d.f.

$$f(x, \alpha, \beta) = \begin{cases} \dfrac{1}{2\beta}, & \alpha - \beta \leq x \leq \alpha + \beta \\ 0 & , \text{elsewhere,} \end{cases}$$

where $\beta > 0$ and $-\infty < \alpha < \infty$ are unknown parameters.

(a) Find the maximum likelihood estimates of α and β.

(b) Find the method of moment estimates of α and β.

7. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from independent exponential distributions with parameters μ and$\lambda\mu$ respectively. Let $X_{(1)} = \min(X_1, \ldots, X_m)$ and $Y_{(1)} = \min(Y_1, \ldots, Y_n)$.

   (a) Find $P(X_1 > Y_1)$.

   (b) Find $P(X_{(1)} > Y_{(1)})$.

   (c) Find the MLE of $P(X_1 > Y_1)$ and hence find the MLE of $P(X_{(1)} > Y_{(1)})$,

8. Suppose that $X|n, \theta$ has a binomial distribution with parameter θ. Suppose we put independent prior distributions on n and θ, with n having Poisson(λ) prior and θ having a Beta(α, β) prior, where α and β are known hyperparameters.

   (a) Prove that the posterior density of θ given $X = x$ and n is Beta($x + \alpha, n - x + \beta$).

   (b) Prove that the posterior probability function of $n + X$ given $X = x$ and θ is Poisson$[(1 - \theta)\lambda]$.

   (c) Suppose $\alpha = \beta = 1$ and $X = 10$, explain in details how you can obtained 100 samples of n's from the **posterior distribution** of n given $X = 10$.

9. Let X and Y be random variables such that $Y|X = x \sim \text{Poisson}(\lambda x)$, and X has density

$$f_X(x) = \frac{\theta^\theta x^{\theta-1} e^{-\theta x}}{\Gamma(\theta)}, \quad x \geq 0.$$

Prove that

(a) $E(Y) = \lambda$ and $Var(Y) = \lambda + \theta\lambda^2$.

(b) Y has density

$$f_Y(y;\lambda) = \frac{\Gamma(\theta + y)\lambda^y \theta^\theta}{\Gamma(\theta) y! (\theta + \lambda)^{\theta+y}}, \quad y = 0, 1, 2, \ldots$$

(c) Let $\theta = 1$. Construct a UMP level α test for $H_0 : \lambda = 1$ versus $H_1 : \lambda > 1$. Fully explain your steps.

10. Let $\{X_1, \ldots, X_m\}$ be a random sample from the normal distribution with mean μ_1 and variance σ_1^2 and $\{Y_1, \ldots, Y_n\}$ a random sample from the normal distribution with mean μ_2 and variance σ_2^2. Assume that the X_i's are independently distributed of the Y_j's. Put: $\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i, \hat{\sigma}_1^2 = \frac{1}{m-1}\sum_{i=1}^{m}(X_i - \bar{X})^2$, and
$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i, \hat{\sigma}_2^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$.

   (a) We can approximate the distribution of $\frac{\hat{\sigma}_1^2}{m} + \frac{\hat{\sigma}_2^2}{n}$ with the distribution of $a\chi_b^2$, where $a > 0$ and $b > 0$ are constants to be determined. How to find the values of a and b? (Note that a and b are functions of $(\sigma_i^2, i = 1, 2)$).

   (b) Define the random variable U by:

$$U = \{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)\}/\sqrt{\frac{\hat{\sigma}_1^2}{m} + \frac{\hat{\sigma}_2^2}{n}}.$$

   Using the above approximation, show that U is distributed as t_b where t_b is a central t random variable with degrees of freedom b.

11. Let $\{X_1, \ldots, X_n\}$ be a random sample from the density $f(x, \theta) = \theta(1-\theta)^x, x = 0, 1, \ldots, \infty; 0 < \theta < 1$.

   (a) Obtain a sufficient and complete statistic for θ.

   (b) Obtain the UMVUE (Uniformly Minimum Varianced Unbiased estimator) of θ. What is the UMVUE for $\phi = 1/\theta$?

   (c) Let the prior distribution of θ be given by:

   $$p(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}, 0 < \theta < 1, a > 0, b > 0.$$

   derive the Bayese estimator of θ and the Bayese estimator of $\phi = 1/\theta$ by assuming squared loss function.

   (d) If the loss function of $\hat{\theta}$ is $l(\hat{\theta}, \theta) = |\hat{\theta} - \theta|$, what is the Bayese estimator of θ?.

12. Let $\{X_1, \ldots, X_n\}$ be a random samples from the probability distribution with density $f(x, \theta, \phi) = \phi(x - \theta)^{\phi - 1}, \theta < x < 1 + \theta, \phi > 0$. Assume that θ is known.

    (a) Derive the level-α UMP (Uniformly Most Powerful) test for testing $H_0 : \phi = 1$ versus $H_1 : \phi > 1$.

    (b) Derive the sampling distribution of your testing statistic under H_0.

    (c) Derive the sampling distribution of your testing statistic under H_1 and hence the power function of your test.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999