Statistics Ph.D. Qualifying Exam: Part I

August 9, 2013

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. $X_1, X_2, \cdots, X_n$ are iid from

$$f(x;\theta) = \frac{2x}{\theta} e^{-x^2/\theta} \quad x > 0 \quad \theta > 0.$$

   (a) Find a complete sufficient statistic for θ

   (b) Find a UMP test of $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$ if one exists.

   (c) Now suppose $X_1, X_2, \cdots, X_n$ are iid from

$$f(x;\theta) = \frac{2x}{\theta^2} e^{-x^2/\theta^2} \quad x > 0$$

      i. Find a complete sufficient statistic for θ

      ii. Find a UMP test of $H_0 : \theta \geq \theta_0$ vs $H_1 : \theta < \theta_0$ if one exists.

2. Suppose that $X_1, X_2, \cdots, X_n$ is an iid sample of size n from a Poisson distribution with mean λ. We wish to estimate $\tau(\lambda) = P(X_1 = 0)$. Consider the following two estimators:

$$T_1 = e^{-\bar{X}}, \qquad T_2 = \frac{1}{n}\sum_{i=1}^{n} I_{(X_i=0)}$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$ and I is the indicator function.

(a) Find the asymptotic distribution of T_1.

(b) Find the asymptotic distribution of T_2.

(c) Which estimator is more efficient in estimatinf θ when a large sample size is available? justify.

3. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = \theta^{-1}x^{(1-\theta)/\theta} \qquad 0 < x < 1$$

where $\theta > 0$.

(a) Show that $T(\boldsymbol{X}) = -2\sum_{i=1}^{n} \log X_i$ is a minimal sufficient statistic for θ

(b) Find the distribution of $Y = -2\log X_i$

(c) Find a two-sided 95% confidence interval for θ based on T

(d) Argue or prove that the expected length of your confidence interval in part (c) converges to zero as $n \to \infty$

4. Let $X_1, X_2, \cdots, X_n$ be a random sample from a distribution with pdf given by

$$f(x;\theta) = \theta^{-c} c x^{c-1} e^{-(x/\theta)^c} \qquad x > 0$$

where $c > 0$ is a known constant.

(a) Find the MLE for θ

(b) Find the UMVUE for θ

(c) Find the uniformly most powerful test of size α for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ where θ_0 is a positive constant.

5. Let X_1 and X_2 be two independent random variables following a $U(0,1)$ distribution. For each of the Y defined below, identify its distribution with its associated parameter values (no derivation of its probability density function is necessary):

   (a) $Y = -\ln(X_1^2 \times X_2^2)$.

   (b) $Y = \ln(X_1)/\ln(X_1 \times X_2)$.

   (c) $Y = \lfloor -\ln(X_1) \rfloor + \lfloor -\ln(X_2) \rfloor$, where $\lfloor x \rfloor$ is the floor function which returns the largest integer $\leq x$.

6. Let (X_1, X_2) be a random sample from an exponential distribution with mean θ. Let $S = \min(X_1, X_2)$ and $R = \max(X_1, X_2) - \min(X_1, X_2)$.

   (a) Find the joint distribution of S and R.

   (b) Find the marginal distributions of S and R.

   (c) Find the distribution of $R/(2S)$.

7. Let X, Y, Z be three random variables with a joint pdf

$$f(x,y,z) = \frac{\Gamma(a+b+c+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} x^{a-1} y^{b-1} z^{c-1} (1-x-y-z)^{d-1},$$

where $0 < x < 1;\quad 0 < y < 1;\quad 0 < z < 1;\quad 0 < x+y+z < 1$, a, b, c, d are positive constants and $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

(a) Derive the marginal distribution of X.

(b) Derive the distribution of $X + Y$.

(c) Derive the conditional distribution of Z given $X = x$ and $Y = y$.

8. Let $X_1, \ldots, X_m$ be a random sample from $N(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_n$ a random sample from $N(\mu_2, c^2\sigma^2)$ where c^2 is a known positive number.

   (a) Obtain maximum likelihood estimators of μ_1, μ_2, and σ^2.

   (b) Derive the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ vs. $H_1 : \mu_1 \neq \mu_2$.

9. Let $X_1, \ldots, X_n$ be continuous independent and identically distributed random variables with distribution function F. Let $Y = \min\{F(X_1), \ldots, F(X_n)\}$

   (a) Find the density of Y.

   (b) Prove that

$$\lim_{n \to \infty} P(nY \leq y) = 1 - \exp(-y).$$

10. Suppose that N is a non-negative integer valued-random variable.

(a) Prove that

$$E(N) = \sum_{n=1}^{\infty} P(N \geq n).$$

(b) Let $\{T_n, n \geq 1\}$ be a sequence of independent Bernoulli random variables such that $P(T_n = 0) = \frac{1}{n}$. Let $N = \min\{n \geq 1 : T_n = 1\}$. Find $E(N)$.

11. Suppose that X is a continuous random variable with a density function with a parameter θ. Suppose a prior density is chosen for θ such that the joint density of X and θ is given by

$$f(x, \theta) = \frac{2\theta e^{-\theta x}}{\pi(1+\theta^2)} \quad x \geq 0, \theta \geq 0.$$

(a) Find the prior density of θ.

(b) Find $f(x|\theta)$.

(c) If $X_1, \ldots, X_n$ is a random sample from the conditional distribution given θ, find the MLE of θ.

(d) Describe a strategy that may be used to find the Bayes estimate of θ.

12. Let $X_1, \ldots, X_m$ be a random sample from an exponential distribution with density

$$f_X(x) = \mu\lambda \exp(-\mu\lambda x), \quad x > 0$$

and let $Y_1, \ldots, Y_n$ be a random sample from an exponential distribution with density

$$f_Y(y) = \lambda \exp(-\lambda y), \quad y > 0.$$

Assume that the X's and Y's are independent. Let $S_X = \sum_{i=1}^m X_i$, $S_Y = \sum_{i=1}^n Y_i$, $R = \frac{S_X}{S_Y}$, and $W = S_Y$.

(a) Write down the joint density of S_X, S_Y.

(b) Find the joint density of R, W.

(c) Find the marginal density of R.

(d) Find the marginal log-likelihood of μ based on R.

(e) Compute the MLE of λ, when μ is fixed.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999