Statistics Ph.D. Qualifying Exam: Part I

August 8, 2014

Student Name: _____

Student UID: _____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with mean $\frac{1}{\theta}$, and let $Y_1, \ldots, Y_n$ be an independent random sample from an exponential distribution with mean $\frac{1}{\rho\theta}$.

   (a) Find the MLE's of θ and ρ.

   (b) Construct a likelihood ratio test for testing $H_0 : \rho = 1$ versus $H_1 : \rho \neq 1$

   (c) Give the exact distribution of your test statistic.

2. Suppose that the single observation X has a negative binomial distribution with parameters θ and r (that is X is the number of independent Bernoulli trials needed to obtain r successes, where each trial has probability of success θ. Assume $0 < \theta < 1$. Let $Y = X - r$

   (a) Show that
   $$P(Y = k|\theta) = \frac{\Gamma(r+k)}{k!\Gamma(r)}\theta^r(1-\theta)^k.$$

   (b) Suppose we put a Beta(α, β) prior distribution on θ. Assuming a squared error loss function, find the Bayes estimators of

      i. θ

      ii. $\eta = \frac{1}{\theta}$

3. Let $X_1, \ldots, X_n$ be independent, identically distributed random variables, with the exponential density

$$f(x;\theta) = \theta e^{-\theta x}, x > 0.$$

Let $\hat{\theta}$ denote the MLE of θ.

(a) Obtain the maximum likelihood estimator $\widehat{\theta\alpha}$, for any given value of α.

(b) What is the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

(c) Show that $\hat{\theta}$ is a biased estimator of θ.

(d) What is the minimum variance unbiased estimator of θ?

4. $X_1, X_2, \ldots, X_{2n+1}$ is a random sample from

$$f_X(x;\beta) = \beta e^{-x\beta} \quad x > 0, \quad \beta > 0$$

(a) Find the median of this distribution.

(b) Find the maximum likelihood estimate (mle) of the median.

(c) Find the bias of the mle of the median.

(d) Find the pdf of the sample median.

5. $f_{XY|Z}(x, y|z) = z + (1 - z)(x + y)$ for $0 < x < 1$ and $0 < y < 1$ and $f_Z(z) \sim U(0, 2)$.
   (a) Find $E(X + Y)$.
   (b) Are X and Y independent?
   (c) Are X and Z independent?
   (d) Find the joint distribution of X and $X + Y$.

6. Let X_1, X_2 be iid $U(0,1)$ and $X_{(1)}$, $X_{(2)}$ be the corresponding order statistics.

   (a) What is $f_{X_{(1)}|X_{(2)}}(x_1|x_2)$, the conditional distribution of $X_{(1)}$ given $X_{(2)} = x_2$?

   (b) What is the distribution of $X_{(2)} - X_{(1)}$?

7. Let $X_1, \cdots, X_n$ be an i.i.d sample from the gamma density

$$f(x;\theta) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x}, \tag{1}$$

where θ is an unknown positive parameter and α is a known positive constant.

(a) Find the maximum likelihood estimator of θ.

(b) What is the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

8. Let X_i's and Y_j's are independent samples from populations with densities

$$f(x, \theta_1) = (\frac{x}{\theta_1^2} e^{-x/\theta_1}), \tag{2}$$

$$f(y, \theta_2) = (\frac{y}{\theta_2^2} e^{-y/\theta_2}), \tag{3}$$

for $x > 0$ and $y > 0$, where $i, j = 1, 2, \cdots, n$, $\theta_1 > 0$, and $\theta_2 > 0$. Find the critical region of the likelihood ratio test of size α of $H_0 : \theta_1 = \theta_2 = \theta$ versus $H_1 : \theta_1 \neq \theta_2$.

9. Let $X_i, i = 1, 2, \cdots, n$ be i.i.d random variables having density $f(x;\theta) = e^{-(x-\theta)}$, for $x > \theta$. Let $Y = \min_{1 \leq j \leq n}(X_i)$ and denote the density of Y by $h(y, \theta)$.

   (a) Show that Y is complete and sufficient for the family of densities $\{h(y,\theta)|\theta \in (-\infty, \infty)\}$

   (b) Find the uniformly minimum variance unbiased estimator (UMVUE) for θ.

10. Let X be a random variable with p.d.f. $f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1, 0 < \theta$. Let $Y = -\ln(X)$.

    (a) Find the moment generating function of Y, $M_Y(t) = E(e^{tY})$.

    (b) Using the m.g.f. found above or direct transformation, find the p.d.f. of $Y = -\ln(X)$.

    (c) Find $E(Y)$, the mean of Y.

    (d) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n with p.d.f. $f(x;\theta)$. Find the maximum likelihood estimator of θ.

    (e) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n with p.d.f. $f(x;\theta)$. Find the method of moments estimator of θ.

11. Let X have a uniform distribution $U(0,1)$, and let the conditional distribution of Y, given that $X = x$, be $U(0,x)$.

    (a) Determine the joint p.d.f. of X and Y, $f(x,y)$.

    (b) Find the marginal p.d.f of Y, $f_2(y)$.

    (c) Compute $E(Y|x)$, the conditional mean of Y, given that $X = x$.

    (d) Compute $E(X|y)$, the conditional mean of X, given that $Y = y$.

12. Let X_1 and X_2 be independent random variables. Assume that X_1 follows a binomial distribution $B(50, 0.4)$ and X_2 follows a Poisson distribution with mean 30, respectively. Let $Y = X_1 + X_2$ be the sum of two random variables.

    (a) Find the moment generating function of Y.

    (b) Approximate $P(X_1 = 20)$.

    (c) Approximate $P(45 \leq Y \leq 55)$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999