Statistics Ph.D. Qualifying Exam: Part II

August 15, 2014

Student Name: ____________________

Student UID: ____________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.
3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)
4. You can use the N(0,1) distribution table as attached.

1. Let Y_{ij}, $(i = 1, 2; j = 1, 2, 3$ be independent random variables such that $Y_{ij} \sim N(\mu_i, i^2\sigma^2)$, for $j = 1, 2, 3$.

   (a) Find the least squares estimators of μ_1 and μ_2.

   (b) Find the Maximum Likelihood estimators of μ_1, μ_2, and σ^2.

   (c) Find an unbiased estimator of σ^2.

   (d) Construct a test statistic for testing $H_0 : (\mu_1, \mu_2) = (a_1\mu, a_2\mu)$ versus $H_1 : (\mu_1, \mu_2) \neq (a_1\mu, a_2\mu)$ where a_1, a_2 are known constants.

2. Suppose that $X_1, \dots, X_n$ is a random sample from a population with density

$$f(x|\theta) = \frac{\theta e^{\theta x}}{e^{\theta} - 1}, \quad 0 < x < 1,$$

where $\theta > 0$.

(a) Construct a uniformly most powerful test of size α for testing

$$H_0 : \theta \leq 1 \text{ versus } H_1 : \theta > 1.$$

(b) Using the Central Limit Theorem, find an approximate rejection region for the UMP test at size $\alpha = 0.05$, and hence approximate the power function of the UMP test.

3. Suppose that $X|n, \theta$ has a binomial distribution with parameter θ. Suppose we put independent prior distributions on n and θ, with n having Poisson(λ) prior and θ having a Beta(α, β) prior, where α and β are known hyperparameters.

   (a) Prove that the posterior density of θ given $X = x$ and n is Beta($x + \alpha, n - x + \beta$).

   (b) Prove that the posterior probability function of $n + X$ given $X = x$ and θ is Poisson$[(1 - \theta)\lambda]$.

   (c) Suppose $\alpha = \beta = 1$ and $X = 10$, explain in details how you can obtained 100 samples of n's from the **posterior distribution** of n given $X = 10$.

4. Let X and Y be random variables such that $Y|X = x \sim \text{Poisson}(\lambda x)$, and X has density

$$f_X(x) = \frac{\theta^\theta x^{\theta-1} e^{-\theta x}}{\Gamma(\theta)}, \quad x \geq 0.$$

(a) Prove that

i. $E(Y) = \lambda$ and $Var(Y) = \lambda + \theta\lambda^2$.

ii. Y has density

$$f_Y(y; \lambda) = \frac{\Gamma(\theta + y)\lambda^y \theta^\theta}{\Gamma(\theta) y! (\theta + \lambda)^{\theta+y}}, \quad y = 0, 1, 2, \ldots$$

(b) Now suppose that $Y_1, \ldots, Y_n$ are independent random variables from the distribution given above, with Y_i having mean λ_i, and $\log(\lambda_i) = \beta z_i$, where z_i's are known covariates, $i = 1, \ldots, n.$, and assume that $\theta = 1$. Write a Fisher scoring algorithm for computing the MLE of β, and discuss its properties.

5. Let $X_1, X_2, \ldots, X_n$ be iid from

$$f_X(x;\theta) = \theta(1+x)^{-(1+\theta)} \quad x > 0 \quad \theta > 0$$

(a) Estimate θ by the method of moments assuming $\theta > 1$.

(b) Find the maximum likelihood estimator (mle) of $\frac{1}{\theta}$.

(c) Find a complete sufficient statistic for θ

(d) Find the Cramer-Rao lower bound for unbiased esimates of $\frac{1}{\theta}$.

(e) Find the UMVUE of $\frac{1}{\theta}$.

6. Let $X_1, \ldots, X_n$ be iid from $f(x; \theta_1) = \theta_1 x^{\theta_1 - 1}$ for $0 < x < 1$ and $Y_1, \ldots, Y_m$ be iid from $f(y; \theta_2) = \theta_2 y^{\theta_2 - 1}$ for $0 < y < 1$. Find the likelihood ratio test for testing $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$.

7. Let $X_1, X_2, \ldots, X_n$ be iid $Poisson(\lambda)$. Let $\bar{X}$ and S^2 be the sample mean and sample variance respectively.

   (a) Prove that $\bar{X}$ is the UMVUE of λ.

   (b) Prove that $E(S^2|\bar{X}) = \bar{X}$ and use this to show that $Var(S^2) > Var(\bar{X})$

8. Suppose that Y_i's are i.i.d random variables with density

$$f(y) = \lambda e^{-\lambda(y-\mu)} I_{(y>\mu)} \tag{1}$$

for $y > 0$, where $\lambda > 0, \mu > 0$ are unknown parameters.

(a) Find the minimal sufficient statistics for (μ, λ). And prove these are complete for $n = 2$.

(b) Suppose that you observe a sample of $n = 2$ variables Y_i, and that you are told that $\mu < 10$. Find a UMVUE for $e^{-\lambda(10-\mu)}$.

9. Let $X_1, \cdots, X_n$ be i.i.d with a common uniform distribution on $[-\theta, \theta]$.

   (a) Find $\hat{\theta}$, the maximum likelihood estimator of θ.

   (b) Prove that $\hat{\theta}$ is consistent.

10. Let $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ be independent normal random variables, let $E(X_i) = \mu_1$, $Var(X_i) = \sigma_1^2$, $E(Y_j) = \mu_2$ and $Var(Y_j) = \sigma_2^2$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$. Assume that both m and n are large.

    (a) Find a confidence interval for $\delta = \mu_1 - \mu_2$, with approximate coverage probability $1 - \alpha$, assuming that the variances are unknown but equal.

    (b) Redo part (a) by assuming that the variances are unknown and possibly unequal.

11. Let X_1 and X_2 be two independent random variables following chi-square distributions with degrees of freedom v_1 and v_2, respectively. Define $Y = \frac{X_1/v_1}{X_2/v_2}$.

(a) Derive the p.d.f. of Y.

(b) Derive the mean of Y.

12. For each of the following pdfs, let $X_1, X_2, \cdots, X_n$ be a random sample from that distribution. In each case, find the UMVUE of θ^r, where $r < n$ is an integer.

   (a) $f(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$.

   (b) $f(x;\theta) = e^{-(x-\theta)}$, $\theta < x$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999