Statistics Ph.D. Qualifying Exam: Part I

August 7, 2015

Student Name: ____________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. If X and Y have joint density given by

$$f_{XY}(x,y) = \frac{y}{(1+x)^4} e^{-y/(1+x)} \quad x > 0, y > 0$$

(a) Find $P\left((X < 1 \cap Y > 0) | X < 2\right)$.

(b) Find $E(Y^r | X = x)$

2. (a) Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. Find an unbiased estimator of σ of the form $\hat{\sigma} = kS$ where S^2 is the sample variance given by $S^2 = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$.

(b) Continuous random variables X and Y have joint density function

$$f_{XY}(x, y) = \exp(-(\theta x + \theta^{-1} y)) \quad x > 0, y > 0, \theta > 0$$

$(X_1, Y_1), \ldots, (X_n, Y_n)$ is a random sample of size n from $f_{XY}(x, y)$. Consider estimating θ by $\hat{\theta} = \left(\dfrac{\bar{Y}}{\bar{X}}\right)^{1/2}$ for $\bar{X} = \dfrac{\sum_{i=1}^{n} X_i}{n}$ and $\bar{Y} = \dfrac{\sum_{i=1}^{n} Y_i}{n}$. Find $E(\hat{\theta})$.

3. Let $X_1, \ldots, X_n$ be a random sample of size n from the following distribution

$$f(x;\theta) = \binom{2}{x}\theta^x(1-\theta)^{2-x} \qquad x = 0, 1, 2$$

(a) Find a sufficient statistic for θ and demonstrate whether it is complete.

(b) Find a maximum likelihood estimator of θ^2

(c) Find a UMVUE of θ^2.

(d) Find the Cramer-Rao lower bound of unbiased estimators of θ^2

4. An epidemiologist gathers data (x_i, Y_i) on each of n randomly chosen noncontiguous and demographically similar cities in the United States, where x_i $(i = 1, 2, \cdots, n)$ is the known population size (in millions of people) in city i, and where Y_i is the random variable denoting the number of people in city i with colon cancer. It is reasonable to assume that Y_i $(i = 1, 2, \cdots, n)$ has a Poisson distribution with mean $E(Y_i) = \theta x_i$, where $\theta (> 0)$ is an unknown parameter, and that $Y_1, Y_2, \cdots, Y_n$ are mutually independent random variables.

   (a) Using the available data (x_i, Y_i), $i = 1, 2, \cdots, n$, construct a UMP test of $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

   (b) If $\sum_{i=1}^{n} x_i = 0.82$, what is the power of this UMP test for rejecting $H_0 : \theta = 1$ versus $H_1 : \theta > 1$ where the nominal Type I error $\alpha = 0.05$ and in reality $\theta = 5$?

5. Suppose we have a sample of size n from a distribution with CDF

$$F(x|\alpha, \beta) = (x/\beta)^{\alpha} I(0 \leq x \leq \beta) + I(x > \beta),$$

where $\alpha > 0$, $\beta > 0$ and $I(\cdot)$ is an indicator function. Find the MLE's of the two parameters.

6. Let $Y_1, Y_2, \cdots, Y_n$ constitute a random sample of size n ($n \geq 2$) from $N(0, \sigma^2)$.

   (a) Develop an explicit expression for an unbiased estimator $\hat{\theta}$ of the unknown parameter $\theta = \sigma^r$ (r a known positive integer) that is a function of a sufficient statistic for θ.

   (b) Derive an explicit expression for the CRLB for the variance of any unbiased estimator of the parameter $\theta = \sigma^r$. Find a particular value of r for which the variance of $\hat{\theta}$ actually achieves the CRLB.

7. Let X be a random variable with p.d.f. $f(x;\theta) = \theta x^{\theta-1}, 0 < x < 1, 0 < \theta$. Let $Y = -\ln(X)$.

   (a) Find the moment generating function of Y, $M_Y(t) = E(e^{tY})$.

   (b) Using the m.g.f. found above or direct transformation, find the p.d.f. of $Y = -\ln(X)$.

   (c) Find $E(Y)$, the mean of Y.

   (d) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n with p.d.f. $f(x;\theta)$. Find the maximum likelihood estimator of θ.

   (e) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n with p.d.f. $f(x;\theta)$. Find the method of moments estimator of θ.

8. Let X_1 and X_2 be two independent random variables, each with a Gamma distribution with p.d.f $f(x) = xe^{-x}, x > 0$. Let us consider

$$Y_1 = \frac{X_1}{X_1 + X_2}, \quad Y_2 = X_1 + X_2.$$

(a) Find the joint pdf of Y_1 and Y_2, $g(y_1, y_2)$.

(b) Find the marginal p.d.f of Y_1, $g_1(y_1)$.

(c) Find the marginal p.d.f of Y_2, $g_2(y_2)$.

(d) Find the distribution of $W = \frac{X_1}{X_2}$.

9. For the hierarchical model

$$Y|\Lambda \sim \text{Poisson}(\Lambda) \quad \text{and} \quad \Lambda \sim \text{Gamma}(\alpha, \beta).$$

(a) Find the marginal distribution of Y.

(b) If α is an integer, show that the distribution of Y is a negative binomial distribution.

(c) Find the moment generating function of Y.

(d) Find the mean and variance of Y.

10. Let $Y_1, \ldots, Y_n$ be independent random variables such that $Y_j \sim N(\theta, \sigma_j^2), j = 1, \ldots, n$. Consider all estimators T, of θ that satisfy the following properties:

   (a) T is an unbiased estimator of of θ.

   (b) T is a linear function of $Y_1, \ldots, Y_n$, that is

$$T = \sum_{j=1}^{n} c_j Y_j,$$

   where $c_1, \ldots, c_n$ are known constant coefficients. Find an estimator T^* such that $Var(T^*) \leq Var(T)$ for all $-\infty < \theta < \infty$.

11. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with mean 1 and let $Y_i = X_{(i)}$ be the ith order statistic $i = 1 \ldots, n$ with $Y_0 = 0$.

   (a) Prove that the random variables $(n+1-i)(Y_i - Y_{i-1}), i = 1, \ldots, n$ are independent and identically distributed. What is the common distribution?

   (b) Calculate $Var(Y_n - Y_1)$.

   (c) Construct a level α chi-squared test for testing $H_0 : F = F_0$ versus $H_1 : F \neq F_0$, where F is a distribution function of a continuous non-negative random variable and F_0 is the distribution function of the exponential random variable with mean 1. State the degree of freedom of your chi-squared statistic.

12. Let $X_1 \ldots, X_{10}$ be a random sample from a population with density

$$f(x|\theta) = e^{-(x-\theta)}, \quad x \geq \theta.$$

You wish to test $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$

(a) Construct the likelihood ratio test for a test of size $\alpha = 0.05$. Write down the rejection region explicitly.

(b) Now consider testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$

(c) Show that the distribution has monotone likelihood ratio property in the sufficient statistic and hence write down an explicit form of the UMP test in terms of tabulated percentiles.

Table of $P(Z < z)$, $Z \sim N(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999