Statistics Ph.D. Qualifying Exam: Part II

August 14, 2015

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Let $X_1, \ldots, X_n$ be a random sample from $f(x;\theta) = \dfrac{1}{\theta} \quad 0 < x < \theta$, for $\theta > 0$.

   (a) Does the family of densities $f(x;\theta)$ have a monotone-likelihood ratio? If so find the corresponding test statistic $T(\boldsymbol{X})$.

   (b) Find a Uniformly Most Powerful (UMP) test of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$

   (c) Find the value of the constant in the rejection region that makes the UMP test a size α test.

2. Suppose that the discrete random variable X_n has a geometric distribution given by

$$f_{X_n}(x_n) = p_n(1 - p_n)^{x_n} \quad x_n = 0, 1, \ldots$$

where $p_n = \dfrac{\lambda}{n}$. Find the limiting value of the moment generating function of $Y_n = X_n/n$ as $n \to \infty$ and use this result to determine the asymptotic distribution of Y_n.

3. Let Z_1, Z_2 be a random sample of size 2 from a standard normal distribution.

   (a) Find the distribution of $\bar{Z}$, the sample mean.

   (b) Find the distribution of $\sum_{i=1}^{2}(Z_i - \bar{Z})^2$.

   (c) Show that $\bar{Z}$ and $\sum_{i=1}^{2}(Z_i - \bar{Z})^2$ are independent.

4. Let $X_1, X_2, \cdots, X_{n_1}$ constitute a random sample of size $n_1 (> 2)$ from a normal parent population with mean 0 and variance θ. Also, let $Y_1, Y_2, \cdots, Y_{n_2}$ constitute a random sample of size $n_2 (> 2)$ from a normal parent population with mean 0 and variance θ^{-1}. The set of random variables $\{X_1, X_2, \cdots, X_{n_1}\}$ is independent of the set of random variables $\{Y_1, Y_2, \cdots, Y_{n_2}\}$, and $\theta\,(> 0)$ is an unknown parameter.

   (a) Derive an explicit expression for $E(\sqrt{L})$ when $L = \sum_{i=1}^{n_1} X_i^2$.

   (b) Using all (n_1+n_2) available observations, derive an explicit expression for an exact $100(1-\alpha)\%$ CI for the unknown parameter θ. If $n_1 = 8$, $n_2 = 5$, $\sum_{i=1}^{8} x_i^2 = 30$, and $\sum_{i=1}^{5} y_i^2 = 15$, compute a 95% confidence interval for θ.

5. Let $Y_1, Y_2, \cdots, Y_n$ constitute a random sample of size n from a $N(0, \sigma^2)$ population. Develop the structure of the rejection region for a uniformly most powerful (UMP) test of $H_0 : \sigma^2 = 1$ versus $H_1 : \sigma^2 > 1$. Then, use this result to find a reasonable value for the smallest sample size (say, n^*) that is needed to provide a power of at least 0.8 for rejecting H_0 in favor of H_1 when $\alpha = 0.05$ and when the actual value of σ^2 is no smaller than 2.0 in value.

6. Consider a sample of size n from Uniform $(\theta, \theta+1)$, $\theta \in (-\infty, \infty)$. Find the minimal sufficient statistic and prove your assertion.

7. Let X_i, $i = 1, 2, \cdots, n$ be a random sample from from the pdf

$$f(x;\theta) = c\theta^2 x^{-3}, \quad 0 < \theta < x < \infty,$$

where c is a constant to be determined.

(a) Find the constant c.

(b) Find the maximum likelihood estimator of θ.

(c) Find the method of moments estimator of θ.

(d) Find the uniformly minimum variance unbiased estimator of θ.

8. Let X_i, $i = 1, 2, \cdots, n$ be iid random variables with $N(\theta, \theta)$ distribution, where $\theta >$ is an unknown parameter.

   (a) Find the MLE of θ, $\hat{\theta}$.

   (b) Find the asymptotic distribution of the MLE $\hat{\theta}$.

9. Let X_i, $i = 1, 2, \cdots, n$ be iid random variables with $N(\theta, 1)$ distribution, where θ is an unknown parameter. Consider testing $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$, where θ_0 is a known fixed constant.

   (a) Derive the maximum likelihood estimator for θ under $H_0 : \theta \leq \theta_0$.

   (b) Show that the likelihood ratio test for $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ is to reject H_0 when

   $$\bar{X} > k,$$

   for some constant k.

   (c) Find the constant k above so that likelihood ratio test is of size α.

   (d) Show that the above likelihood ratio test is a UMP test.

10. Let $Y_1, \ldots, Y_n$ be independent random variables with $Y_i \sim \mathcal{P}_o(e^{\beta_0+\beta_1 x_i})$, where $-\infty < \beta_0, \beta_1 < \infty$, and $x_1, \ldots, x_n$ are fixed covariates.

   (a) Derive the likelihood equations for calculating the MLE's of β_0 and β_1.

   (b) Fully describe how you will actually use these equations to compute these MLE's.

11. Suppose that $X|n,\theta$ has a binomial distribution with parameter θ. Suppose we put independent prior distributions on n and θ, with n having Poisson(λ) prior and θ having a Beta(α, β) prior, where α and β are known hyper parameters.

    (a) Prove that the posterior density of θ given $X = x$ and n is Beta($x + \alpha, n - x + \beta$).

    (b) Prove that the posterior probability function of $n + X$ given $X = x$ and θ is Poisson$[(1 - \theta)\lambda]$.

    (c) Suppose $\alpha = \beta = 1$ and $X = 10$, explain in details how you can obtained 100 samples of n's from the **posterior distribution** of n given $X = 10$.

12. Let X and Y be random variables such that $Y|X = x \sim$ Poisson(λx), and X has density

$$f_X(x) = \frac{\theta^\theta x^{\theta-1} e^{-\theta x}}{\Gamma(\theta)}, \quad x \geq 0.$$

(a) Prove that

i. $E(Y) = \lambda$ and $Var(Y) = \lambda + \theta\lambda^2$.

ii. Y has density

$$f_Y(y; \lambda) = \frac{\Gamma(\theta + y)\lambda^y \theta^\theta}{\Gamma(\theta) y! (\theta + \lambda)^{\theta+y}}, \quad y = 0, 1, 2, \ldots$$

(b) Now suppose that $Y_1, \ldots, Y_n$ are independent random variables from the distribution given above, with Y_i having mean λ_i, and $\log(\lambda_i) = \beta z_i$, where z_i's are known covariates, $i = 1, \ldots, n.$, and assume that $\theta = 1$. Write a Fisher scoring algorithm for computing the MLE of β, and discuss its properties.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999