Statistics Ph.D. Qualifying Exam: Part II

August 12, 2016

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Suppose X is one observation from a Binomial (5,θ) distribution where $0 < \theta < 1$.

   (a) If $\pi(\theta)$, the prior distribution for θ, is a Beta(1,1) distribution, find the posterior distribution for θ?

   (b) Suppose we wish to test $H_0 : \theta = 1/2$ versus $H_1 : \theta = 3/4$. What is the rejection region corresponding to the uniformly most powerful (UMP) level-α test? Justify your answer.

2. Assume that X_1 and X_2 are two random samples from a Poisson distribution with $f(x) = \frac{\theta^x exp(-\theta)}{x!}$ for x=0,1,2,$\cdots$ and zero otherwise, where $\theta > 0$.

   (a) Find the moment generating function of X_1.

   (b) What is the probability distribution of $X_1 + X_2$? Justify your response.

   (c) Find the conditional distribution of X_1 given that $X_1 + X_2 = t$.

3. Let $f(x|\theta) = \frac{e^{x-\theta}}{(1+e^{x-\theta})^2}$ (Logistic pdf).

   (a) Does the family (i.e. logistic distribution family) have an MLR?

   (b) Based on one observation, find the UMP size α test for $H_0 : \theta \leq 0$ versus $\theta > 0$.

4. A random number N of fair dice are tossed where $P(N = n) = \theta(1 - \theta)^{n-1}$ $n = 1, 2, \ldots$ and $0 < \theta < 1$. Find the probability that the largest number shown by any of the dice does not exceed k for $k = 1, 2, \ldots, 6$.

5. U_1 and U_2 are iid $U(0,1)$. $U_{(1)}$ and $U_{(2)}$ are the corresponding order statistics.

(a) Find the conditional distribution of $U_{(1)}$ given $U_{(2)} = u_2$ for $0 < u_2 < 1$.

(b) What is the distribution of $U_{(2)} - U_{(1)}$.

6. Suppose $Y_1, \ldots, Y_n$ is a random sample of size n from density function

$$f_Y(y;\theta) = \theta y^{(\theta-1)} \quad 0 < y < 1 \quad \theta > 0$$

Consider random variable $U = nY_{(1)}^{\theta}$ where $Y_{(1)} = \min\{Y_1, \ldots, Y_n\}$. Show that as $n \to \infty$, the distribution of U tends to an exponential distribution.

7. Let $\{(Y_i, x_i); i = 1, \ldots, n\}$ satisfy the regression model

$$Y_i = \beta x_i + \epsilon_i,$$

where $E(\epsilon_i) = 0$, $\text{Var}(\epsilon_i) = \sigma^2$ and ϵ_i's are independent.

(a) Find $\hat{\beta}$ the least squares estimator of β.

(b) Let $\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \hat{\beta})^2$. Is $\hat{\sigma}^2$ an unbiased estimator of σ^2? (Give a proof of your answer.)

(c) Are $\hat{\beta}$ and $\hat{\sigma}^2$ independent? (Give a proof of your answer.)

(d) Construct a test with level of significance α for testing $H_0 : \beta = 0$ versus $H_1 : \beta \neq 0$. and state the properties of your test.

(e) Prove that, with an appropriate choice of scaling factor a_n,

$$a_n \frac{\hat{\beta} - \beta}{\hat{\sigma}}$$

converge in distribution to a $N(0, 1)$ distribution.

8. Let $X_1, \ldots, X_n$ be a random sample from Uniform $(0, \theta)$, where $\theta > 0$. Suppose that we put the Gamma prior density

$$\pi(\theta|\gamma) = \frac{\theta^n}{\gamma^{n+1}\Gamma(n+1)} e^{-\frac{\theta}{\gamma}}, \quad \theta > 0.$$

on θ. Let $Y = X_{(n)} = \max\{X_1, \ldots, X_n\}$

(a) Find the density $g(y|\theta)$ of Y given θ.

(b) Find the marginal (unconditional) density of Y.

(c) Find the posterior density $q(\theta|Y = y)$ of θ given Y.

(d) Find the Bayes estimator of θ using squared error loss function.

(e) Find the maximum likelihood estimator of θ.

(f) Compare the mean squared errors of the Bayes estimator and the MLE.

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with density

$$f(x|\theta, \lambda) = \begin{cases} \lambda e^{-\lambda(x-\theta)}, & x \geq \theta \\ 0 \ , & \text{otherwise} \end{cases}$$

(a) Find a jointly complete sufficient statistic for (θ, λ).

(b) Find UMUVE of θ.

(c) Find UMUVE of λ.

10. Let $X = R\cos(\theta)$ and $Y = R\sin(\theta)$, where $\theta \sim U(0, 2\pi)$ and R is a positive random variable.

(a) Find the distribution of X/Y.

(b) Suppose that $R = \sqrt{W}$, where W is a random variable following an exponential distribution with mean c. Find the marginal distribution of X and Y.

11. Let X and Y be two independent random variables with $X \sim Exp(\mu_X)$ and $Y \sim Exp(\mu_Y)$. Suppose that we cannot observe X or Y directly. Instead, we observe the random variables Z and W, where $Z = \min(X, Y)$ and

$$W = \begin{cases} 1, & \text{if } Z = X; \\ 0, & \text{if } Z = Y. \end{cases}$$

(a) Find the marginal distribution of Z.

(b) Find the marginal distribution of W.

(c) Find the joint distribution of Z and W.

(d) Prove that Z and W are independent.

12. Let $\boldsymbol{X} = (X_1, \ldots, X_K)$ be a multinomial vector with parameters $m, K, \boldsymbol{\theta}$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_K)$. That is,

$$f(\boldsymbol{x}|\boldsymbol{\theta}) = \frac{m!}{\prod_{i=1}^{m} x_i!} \prod_{i=1}^{K} \theta_i^{x_i},$$

where $x_K = m - \sum_{i=1}^{K-1} x_i$ and $\sum_{i=1}^{K} \theta_i = 1$. Suppose $\boldsymbol{\theta}$ has Dirichlet prior density

$$\pi(\boldsymbol{\theta}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_K)}{\prod_{j=1}^{K} \Gamma(\alpha_j)} \theta_1^{\alpha_1 - 1} \cdots \theta_K^{\alpha_K - 1},$$

(a) Find the posterior distribution of $\boldsymbol{\theta}$.

(b) Using the loss function $L(\boldsymbol{\theta}, \boldsymbol{d}) = \sum_{i=1}^{K} (\theta_i - d_i)^2$, show that the Bayes estimator of $\boldsymbol{\theta}$ is given by

$$\boldsymbol{d}_0(\boldsymbol{X}) = E(\boldsymbol{\theta}|\boldsymbol{X}) = \frac{\boldsymbol{\alpha} + \boldsymbol{X}}{\sum_{i=1}^{K} \alpha_i + m}.$$

(c) Suppose that $\boldsymbol{S}_n = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ is a random sample of size n from the multinomial distribution. Let $\boldsymbol{d}(\boldsymbol{S}_n)$ be the Bayes estimator of $\boldsymbol{\theta}$ based on $\boldsymbol{S}_n$.

i. Calculate $\boldsymbol{d}(\boldsymbol{S}_n)$.

ii. What is the limiting distribution of $\boldsymbol{d}(\boldsymbol{S}_n)$ as $n \to \infty$?

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999