Statistics Ph.D. Qualifying Exam: Part I

August 4, 2017

Student Name: ____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Derive the probabilities below - you must show your work.

   (a) If X, Y, and Z are independent identically distributed uniform random variables on the interval $(0, \theta)$, i.e. *iid* $U(0, \theta)$, find the probability that X is less than the minimum of Y and Z.

   (b) If X_1, X_2, and X_3 are independently distributed exponential random variables with $E(X_i) = 1/\theta_i$, find the probability that X_1 is less than the minimum of X_2 and X_3.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from

$$f_X(x;\alpha,\beta) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x} \quad x > 0; \alpha > 0, \beta > 0$$

where β is a known positive constant and α is an unknown parameter.

(a) Find a sufficient statistic for the unknown parameter α.

(b) Let $\hat{\alpha} = \dfrac{\beta}{\bar{X}}$. Find $E(\hat{\alpha})$ and $Var(\hat{\alpha})$. Is $\hat{\alpha}$ a consistent estimator of α?

(c) If $\beta = 2$, $n = 50$ and $\bar{x} = 3$ use the central limit theorem to develop an appropriate large-sample approximate 95% confidence interval for α.

3. Let Y_1 and Y_2 be a random sample of size $n = 2$ from the distribution

$$f_Y(y) = \theta y^{\theta-1} \quad 0 < y < 1, \theta > 0$$

(a) Find the most powerful test of $H_0 : \theta = 1$ vs $H_1 : \theta = 2$.

(b) Find the pdf of $U = Y_1 Y_2$.

(c) Write an integral needed to determine the cut-off value so that the most powerful test is of size α.

(d) Write an integral to find the power of the most powerful test

4. A trial may result in three possible outcomes, A with probability p_1, B with probability p_2 and C with probability $1-p_1-p_2$. In n independent trials, A was observed n_1 times and B was observed n_2 times.

   (a) Show that the likelihood ratio test of the null hypothesis $H_0 : p_1 = p_2$ versus the alternative $H_1 : p_1 \neq p_2$ accepts H_0 if $n_1 = n_2 > 0$.

   (b) Find an approximate critical region for testing H_0 of asymptotic level 0.05 for large n.

5. Let $X_1, \cdots, X_n$ be a random sample from the exponential distribution with density $f(x, \lambda) = \frac{1}{\lambda} exp(-x/\lambda)$, $x > 0$.

   (a) Find the UMP test for testing $H_0 : \lambda = 7$ versus $H_1 : \lambda < 7$ at level α.

   (b) Find a $1 - \alpha$ confidence interval for λ by inverting the UMP test.

   (c) Find the expected length of the interval in part (b).

6. Let the random variable Y denote the number of Lyme disease cases that develop in the state of NC during any one calendar year. The event $Y = 0$ is not observable since the observational apparatus (i.e. diagnosis) is activated only when $Y > 0$. Since Lyme diseases is a rare disease, it seems appropriate to model the distribution of Y by the zero-truncated Poisson distribution (ZTPD)

$$P_Y(y) = \frac{(e^\theta - 1)^{-1}\theta^y}{y!}, y = 1, 2, \cdots, \infty$$

where $\theta(> 0)$ is called the "incidence parameter."

(a) Find an explicit expression for $\Phi(t) = E\left[(t+1)^Y\right]$.

(b) Use $\Phi(t)$ to show that

$$E(Y) = \frac{\theta e^\theta}{e^\theta - 1}$$

and

$$Var(Y) = \frac{\theta e^\theta(e^\theta - \theta - 1)}{(e^\theta - 1)^2}.$$

(c) To lower the incidence of Lyme disease in NC, the state health department mounts a vigorous media campaign to educate NC residents about all aspects of Lyme disease (including information about preventing and dealing with tick bites, using protective measures such as clothing and insect repellents, recognizing symptoms of Lyme disease, treating Lyme disease, etc.) Assume that this media campaign has the desired effect of lowering θ to $\pi\theta$, where $0 < \pi < 1$. Let Z be the number of Lyme disease cases occurring during a 1-year period after the media campaign is over. Assume that

$$P_Z(z) = \frac{(\pi\theta)^z e^{-\pi\theta}}{z!}, z = 0, 1, \cdots, \infty,$$

and that Y and Z are independent random variables. There is interest in the random variable $X = Y + Z$, the total number of Lyme disease cases that occur altogether (namely, 1 year before and 1 year after the media campaign). Find an explicit expression for $P(X = x)$.

7. Let X be a discrete random variable with probability density function

$$f(x) = \frac{e^\theta}{(1+e^\theta)^{x+1}}, \quad x = 0, 1, \ldots,$$

where θ is a real parameter.

(a) Find $\Pr(X \text{ is odd})$.

(b) Suppose that X and Y are independent, each with probability function $f(x)$. Find $\Pr(X = Y)$.

(c) Let X and Y be as defined in (b). Find the probability function of $Z = X + Y$.

8. Suppose that $Y \sim N(\theta, 2)$ and consider the following estimates of θ^2:

$$T_1 = Y^2; \qquad T_2 = Y^2 - 2.$$

(a) Show that T_2 is unbiased and has a smaller mean squared error than T_1.

(b) Which of the two do you prefer? And Why?

9. Suppose that $X_1, \ldots, X_n$ are independently and identically distributed with a density function

$$f(x|\mu) = \begin{cases} e^{-(x-\mu)}, & x \geq \mu \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in the testing problem

$$H_0 : \mu = 0 \text{ versus } H_1 : \mu = 1.$$

Consider the critical region given by $\min(X_1, \ldots, X_n) > C$.

(a) For fixed n, find C so that type I error is 0.1.

(b) For fixed n, find C so that type II error is 0.1.

(c) Find a set of values for C and n so that both types of errors do not exceed 0.1.

10. Let U be a Uniform $(0,1)$ random variable. Let λ a constant, such that $0 < \lambda < 1$, and let V be a random variable with support $(0,1)$ that is independent of U.

   (a) Prove that $\min\left(\frac{U}{\lambda}, \frac{1-U}{1-\lambda}\right)$ has a Uniform$(0,1)$ distribution.

   (b) Find $P(\min\left(\frac{U}{V}, \frac{1-U}{1-V}\right) > .5)$.

   (c) If $X \sim N(0,4)$ and $Z \sim N(0,1)$ with distribution function Φ. Find a relation between the distribution functions of $2\min(\Phi(X),(1-\Phi(X))$ and $2\min(\Phi(Z),(1-\Phi(Z))$.

11. Let $X_1, X_2, \ldots,$ be a sequence of independent identically distributed exponential random variables with parameter λ. Let N be a geometric random variable with parameter p. Assume that the X's and N are independent. Find the following

(a) $E(\frac{\sum_{i=1}^{N}}{N} X_i)$

(b) $P(X_{(N)} > a)$

(c) $E(X_{(N)})$

12. Let $X_1, \ldots, X_n$ be a independent random variables such that $X_i \sim Normal(\theta a_i, 1)$.

(a) Find the maximum likelihood estimator of θ.

(b) Use this MLE of θ to construct a $100(1-\alpha)$ confidence interval for θ.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999