Statistics Ph.D. Qualifying Exam: Part I

August 3, 2018

Student Name: _____

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Let $X_1, \ldots, X_n$ be a sample from a distribution with pdf $f(x;\theta) = \theta x^{\theta-1} \quad 0 < x < 1$.

   (a) Find the mle of $\dfrac{1}{\theta}$. Is it unbiased?

   (b) Find the Cramer-Rao lower bound for unbiased estimates of $\dfrac{1}{\theta}$ and check whether the mle achieves the lower bound.

   (c) Find an unbiased estimator of $\dfrac{\theta}{\theta+1}$.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & x = 1, 2, \ldots, \theta, \theta \in \{1, 2, \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

(a) Find a minimal sufficient statistic for θ.

(b) Show that the minimal sufficient statistic is complete.

(c) Show that $2\bar{X} - 1$ is an unbiased estimator of θ.

(d) Is $2\bar{X} - 1$ a UMVUE of θ? Justify your response.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(0, \sigma^2)$.

   (a) Find a uniformly most powerful sizeα test (UMP) for $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_1 : \sigma^2 > \sigma_0^2$.

   (b) Write an expression for finding the rejection region of the UMP test.

4. Suppose that for $i = 1, \ldots, n$,

$$Y_i = \beta x_i + \epsilon_i,$$

where x_i's are fixed covariates, and ϵ_i are independent $N(0, \sigma^2)$ random variables.

(a) Find

i. Method of moments estimators of β and σ^2.

ii. Maximum likelihood estimators of β and σ^2.

(b) Which of these estimators are unbiased? (give a proof)

(c) Which of these estimators are minimum variance unbiased? (give a proof)

5. Suppose that the weights of birds in a flock are normally distributed with mean 6.8 lbs and standard deviation 1.6 lbs. A random sample of 49 birds is selected. Let $\bar{X}$ be the sample mean. If birds weighing more that 10 lbs are tagged, what is the probability that

   (a) at most 48 birds are tagged?

   (b) more than one bird is tagged?

   (c) if a random sample of 1000 birds is selected , approximately what is the probability that at least five birds are tagged?

6. Let $X_1, X_2, \ldots,$ be a sequence of independent exponential random variables with mean μ, and let N be a Poisson (λ) random variable that is independent of the X's. If $X_{(1;N)} = \min\{X_1, \ldots, X_N\}$, Find

   (a) $P(X_{(1;N)} > a)$

   (b) $E[X_{(1;N)}]$

   (c) $E[X_{(1;N)} | X_{(1;N)} > a]$

7. Consider a random sample of size n from a distribution with pdf

$$f(x;\theta) = \frac{(ln\theta)^x}{\theta x!}, x = 0, 1, \cdots; \theta > 1 \tag{1}$$

and zero otherwise.

(a) Find a complete sufficient statistic for θ.

(b) Find the MLE of θ.

(c) Find the CRLB for θ.

(d) Find the UMVUE of $ln\theta$.

(e) Find the UMVUE of $(ln\theta)^2$.

(f) Find the CRLB for $(ln\theta)^2$.

8. Consider a random sample of size n from a uniform distribution, $X_i \sim UNIF(0, \theta)$, $\theta > 0$, and let $X_{n:n}$ be the largest order statistic.

   (a) Find the probability that the random interval $(X_{n:n}, 2X_{n:n})$ contains θ.

   (b) Find the constant c such that $(x_{n:n}, cx_{n:n})$ is a $100(1-\alpha)\%$ confidence interval for θ.

9. From the set $\{1, 2, 3, ..., n\}$, k distinct integers are selected at random and arranged in numerical order (from lowest to highest). Let $P(i, r, k, n)$ denote the probability that integer i is in position r. For example, observe that $P(1, 2, k, n) = 0$, as it is impossible for the number 1 to be in the second position after ordering.

   We consider $r = 3$, $k = 5$ and $n = 7$ and let X denote the value at the second position.

   (a) Compute $P(2, 3, 5, 7)$.

   (b) Compute $P(4, 3, 5, 7)$.

   (c) What is the range (possible values) of X ?

   (d) What is p.m.f. of X, $f(x) = Pr(X = x)$?

   (e) What is $E(X)$?

10. Let X and Y be independent uniform(0,1) random variables.

    (a) Find the pdf of $X + Y$.

    (b) Find the pdf of $X - Y$.

    (c) Find the pdf of XY.

    (d) Find the pdf of X/Y.

11. For each of the following pdfs/pmfs, let X_i, $i = 1, 2, \cdots, n$ be a random sample from that distribution. In each case, find the best unbiased estimator of θ^r, where $r < n$.

(a) $f(x;\theta) = \frac{1}{\theta}, \quad 0 < x < \theta.$

(b) $f(x;\theta) = e^{-(x-\theta)}, \quad \theta < x < \infty.$

(c) $f(x;\theta) = \theta^x (1-\theta)^{1-x}, \quad x = 0, 1; 0 < \theta < 1.$

12. Let X_i, $i = 1, 2, \cdots, n$ be iid random variables with $N(\theta, 1)$ distribution, where θ is an unknown parameter. Consider testing $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$, where θ_0 is a known fixed constant.

    (a) Derive the maximum likelihood estimator for θ under $H_0 : \theta \leq \theta_0$.

    (b) Show that the likelihood ratio test for $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ is to reject H_0 when

    $$\bar{X} > k,$$

    for some constant k.

    (c) Find the constant k above so that likelihood ratio test is of size α.

    (d) Show that the above likelihood ratio test is a UMP test.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999