Statistics Ph.D. Qualifying Exam: Part II

August 10, 2018

Student Name: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. The number of breakdowns, X, per day for a certain machine is a Poisson random variable with unknown mean θ. Let $X_1, X_2, \ldots, X_n$ be the number of breakdowns for n independently selected days.

   (a) Derive the distribution of $Y = \sum_{i=1}^{n} X_i$.

   (b) Find the mle of θ.

   (c) Show that the mle of θ is a consistent estimator of θ.

   (d) Show that the mle of θ is an efficient estimator of θ.

   (e) The daily cost of repairing the breakdowns is given by $Z = 3X^2$. Find the mle of $E(Z)$.

   (f) Find the asymptotic distribution of the mle of θ.

   (g) Find a transformation $g(\cdot)$ that satisfies

$$\sqrt{n}\left(g(\bar{X}) - g(\theta)\right) \rightarrow N(0,1)$$

2. $X_1, X_2, \ldots, X_n$ is a random sample of Bernoulli random variables with parameter p. The odds of a success occurring are $\dfrac{p}{1-p}$.

   (a) Find the mle of $\dfrac{p}{1-p}$.

   (b) Use the delta method to find the asymptotic distribution of the mle of the odds of success.

   (c) Suppose that drivers who don't wear seat belts are three times more likely to die in a car accident then those who do wear a seat belt. Suppose the odds of a randomly selected driver wearing a seat belt are 9 (i.e. 9 to 1 in favor of wearing a seat belt). What are the odds that the driver was wearing a seat belt given that the driver died in a car crash?

3. Let Y_1 and Y_2 be a random sample of size $n = 2$ from pdf

$$f_Y(y;\theta) = \theta y^{\theta-1} \quad 0 < y < 1,\ \theta > 0.$$

(a) Find the pdf of $U = Y_1 Y_2$.

(b) Find the rejection region of the most powerful (MP) test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ in terms of U.

(c) Set up an appropriate integral needed to determine the value of k, the constant that defines the rejection region of the MP test so that the probability of a Type I error is α.

(d) Set up the appropriate integral needed to find the *power* of the MP test.

4. Let $X_1, X_2, \ldots$ be a sequence of independent $\mathcal{E}(1)$ random variables. Let N be a $\mathcal{G}\text{eom}(p)$, and assume that N is independent of the X's. Consider the order statistics $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(N)}$.

   (a) Find $E(\frac{\sum_{i=1}^{N} X_{(i)}}{N})$.

   (b) Find $\text{var}(\frac{\sum_{i=1}^{N} X_{(i)}}{N})$.

   (c) Find $P(X_{(N)} \leq x)$.

   (d) Find $E(X_{(N)})$.

5. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x|\theta) = \begin{cases} \left(\frac{\theta}{x}\right)^{\theta+1} & , \quad \text{if} \quad x > \theta \\ 0 & , \quad \text{if otherwise,} \end{cases}$$

where $\theta > 4$.

(a) Find the maximum likelihood estimator (MLE) of the median of this distribution.

(b) Is the above MLE a minimal sufficient statistics? Explain fully.

6. Let $X_1, \ldots, X_n$ be a random sample from $\mathcal{P}\text{oisson}(\theta)$. Let

$$d(\mathbf{X}) = \begin{cases} 1 & , \quad \text{if} \quad X_1 = 0, X_2 = 0 \\ 0 & , \quad \text{if otherwise,} \end{cases}$$

(a) Show that $d(\mathbf{X})$ is an unbiased estimator of $e^{-2\theta}$.

(b) Find the Cramer-Rao lower bound for all unbiased estimator of $e^{-2\theta}$, and show that $d(\mathbf{X})$ does not attain the Cramer-Rao lower bound.

(c) Find the UMVUE estimator of $e^{-2\theta}$. Does this estimator attain the Cramer-Rao lower bound?

7. Let Y be a random variable from negative binomial distribution with pmf

$$P(Y = y) = \binom{m+y-1}{m-1} p^m (1-p)^y, y = 0, 1, \cdots \tag{1}$$

(a) Show that the best unbiased estimator of p is given by $\delta^*(Y) = (m-1)/(Y+m-1)$.

(b) Show that the information contained in Y about p is $I(p) = m/p^2(1-p)$.

(c) Show that $var\delta^* > 1/I(p)$.

8. Consider a random sample $X_1, X_2, \cdots, X_n$ from a distribution with pdf

(a) $f(x, \theta) = [\theta/(\theta + 1)]^x/(\theta + 1)$ if $x = 0, 1, \cdots$ where $\theta > 0$. Find a UMP test of $H_0 : \theta = \theta_0$ against $H_a : \theta > \theta_0$.

(b) $f(x, \theta) = 1/\theta$ if $0 \leq x \leq \theta$ and zero otherwise. Derive the GLR (generalized likelihood ratio) test of $H_0 : \theta = \theta_0$ against $H_a : \theta \neq \theta_0$.

9. Let X and Y be two independent standard Normal random variables.

   (a) Determine $E(XY|X + 2Y)$.

   (b) Determine the distribution of $\frac{X-Y}{X+Y}$.

10. Let $X_1, X_2, ...X_n$ be a random sample from a population with probability function

$$f(x; \theta) = \theta(1-\theta)^x, x = 0, 1, 2, 3, ...$$

where θ is an unknown parameter in $(0, 1)$, under the classical approach. One can also use a Bayesian approach by assuming θ has a prior distribution $\theta \sim U(0, 1)$.

(a) Compute the Cramer-Rao Lower Bound for unbiased estimators of θ.

(b) Find the UMVUE for θ, if possible.

(c) If the loss function $L(\theta, a) = (\theta - a)^2$, find the Bayes estimator of θ.

(d) If the loss function $L(\theta, a) = \frac{(\theta-a)^2}{\theta(1-\theta)}$, find the Bayes estimator of θ.

11. Let $X_1, \ldots, X_n$ be a random sample of size n from a population with density

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}, \quad x > 0, \theta > 0.$$

We wish to estimate $\tau = P(X_1 > 1) = e^{-1/\theta}$.

(a) Compute the Cramer-Rao lower bound for the variance of unbiased estimator of τ.

(b) Find the maximum likelihood estimator of τ.

(c) Find the uniformly minimum variance unbiased (UMVU) estimator of τ.

12. Let $X_1, \ldots, X_n$ be a random sample of size n from a population with density

$$f(x; \theta, \mu) = \frac{1}{3\theta} e^{-x/\theta} + \frac{2}{3\mu} e^{-x/\mu}, \quad x > 0, \theta > 0, \mu > 0.$$

(a) Use Method of Moments to find estimators of θ and μ.

(b) Find the asymptotic distributions of the estimators found above.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999