Statistics Ph.D. Qualifying Exam: Part I

August 2, 2019

Student Name: ____________________

Student UID: ____________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. X and Y have joint density given by

$$f(x,y) = \begin{cases} 2 & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) If $Z = -\log X$, what is the density of Z?

(b) If $Z = -\log X$ and $W = X + Y$, what is the joint density of Z and W?

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf

$$f(x|\theta) = \theta^{-c} c x^{c-1} e^{-(x/\theta)^c} \qquad x > 0$$

where $c > 0$ is a known constant.

(a) Find the maximum likelihood estimator (MLE) of θ.

(b) Find the UMVUE of θ.

(c) Find the uniformly most powerful (UMP) test of size α for testing

$$H_0 : \theta \leq \theta_0 \ \text{ vs } \ H_1 : \theta > \theta_0.$$

3. Suppose that $X_1, X_2, \ldots, X_n$ is an iid sample, each with probability p of being distributed uniformly over $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and with probability $1 - p$ of being distributed uniformly over $(0, 1)$.

   (a) Find the cumulative distribution function (cdf) and the probability density function (pdf) of X_1.

   (b) Find the maximum likelihood estimator (MLE) of p.

   (c) Find the method of moments (MOM) estimator of p.

   (d) Which of the two estimators (MLE or MOM) is more efficient? Justify your response.

4. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x|\theta) = \begin{cases} \left(\frac{\theta}{x}\right)^{\theta+1} & , \quad \text{if} \quad x > \theta \\ 0 & , \quad \text{if otherwise,} \end{cases}$$

where $\theta > 4$.

(a) Find the maximum likelihood estimator (MLE) of θ^4.

(b) Is the above MLE a minimal sufficient statistics? Explain fully.

5. In order to decide the appropriate amount to charge as premium, insurance companies often use the exponential principle defined as follows: If X is the random amount that it will have to pay in claims, then the premium charged by the insurance company should be

$$P = \frac{1}{a}\ln(E[e^{aX}]),$$

where $a > 0$ is a fixed specified constant. Suppose that an insurance company assumes that X has an exponential distribution with parameter θ.

(a) Find P.

(b) An insurance company wishes to find a maximum likelihood estimator $\hat{P}$ of P, by taking a random sample $X_1, \ldots, X_n$ from a large set of previous payments. Find $\hat{P}$.

6. Let $Y_1, \ldots, Y_n$ be a random sample from Poisson(θ). Let $\bar{Y} = \sum_{i=1}^{n} Y_i$ be the sample mean and $S^2 = \sum_{i=1}^{n} (Y_i - \bar{Y})^2/(n-1)$ be the sample variance.

   (a) Prove that $\bar{Y}$ is a complete and sufficient statistic for θ.

   (b) Prove that both $\bar{Y}$ and S^2 are unbiased estimators of θ.

   (c) Prove that $E(S^2|\bar{Y}) = \bar{Y}$.

   (d) Prove that $\mathrm{Var}(S^2) > Var(\bar{Y})$.

   (e) Is S^2 an admissible estimator of θ?

7. Let X and Y be independent uniform(-1,1) random variables.

   (a) Find the pdf of XY.

   (b) Find the pdf of X/Y.

8. Let X_1 and X_2 be independently and identically distributed from a geometric distribution with probability mass function given by

$$P(X = x) = p(1-p)^{x-1}, \text{ where } x = 1, 2, 3, \ldots.$$

(a) Find the UMVUE for p.

(b) Find the UMVUE for p^2.

9. Let X_1, X_2, X_3 be random variables.

   (a) If $X_1 \sim Poisson(\lambda),\ X_2 \sim Poisson(\mu)$, and X_1, X_2 are independent find $P(X_1 = k|X_1 + X_2 = 2k)$

   (b) If $(X_1, X_2, X_3) \sim \text{Trinomial}(n, p_1, p_2, p_3)$, that is,

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3},$$

   where $p_2 = 1 - p_1 - p_2,\ X_3 = n - X_1 - X_2$, find $P(X_1 = k|X_1 + X_2 = m)$

10. Let the random variable Y denote the number of Lyme disease cases that develop in the state of NC during any one calendar year. The event $Y = 0$ is not observable since the observational apparatus (i.e. diagnosis) is activated only when $Y > 0$. Since Lyme diseases is a rare disease, it seems appropriate to model the distribution of Y by the zero-truncated Poisson distribution (ZTPD)

$$P_Y(y) = \frac{(e^{\theta} - 1)^{-1}\theta^y}{y!}, y = 1, 2, \cdots, \infty$$

where $\theta(> 0)$ is called the "incidence parameter."

(a) Find an explicit expression for $\Phi(t) = E\left[(t+1)^Y\right]$.

(b) Use $\Phi(t)$ to show that

$$E(Y) = \frac{\theta e^{\theta}}{e^{\theta} - 1}$$

and

$$Var(Y) = \frac{\theta e^{\theta}(e^{\theta} - \theta - 1)}{(e^{\theta} - 1)^2}$$

11. Let X be a discrete random variable with probability density function

$$f(x) = \frac{e^{\theta}}{(1+e^{\theta})^{x+1}}, \quad x = 0, 1, \ldots,$$

where θ is a real parameter.

(a) Find $\Pr(X \text{ is odd})$.

(b) Suppose that X and Y are independent, each with probability function $f(x)$. Find $\Pr(X = Y)$.

(c) Let X and Y be as defined in (b). Find the probability function of $Z = X + Y$.

12. Suppose that $X_1, \ldots, X_n$ are independently and identically distributed with a density function

$$f(x|\mu) = \begin{cases} e^{-(x-\mu)}, & x \geq \mu \\ 0, & \text{otherwise.} \end{cases}$$

We are interested in the testing problem

$$H_0 : \mu = 0 \text{ versus } H_1 : \mu = 1.$$

Consider the critical region given by $\min(X_1, \ldots, X_n) > C$.

(a) For fixed n, find C so that type I error is 0.05.

(b) For fixed n, find C so that type II error is 0.05.

(c) Find a set of values for C and n so that both types of errors do not exceed 0.05.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999