Statistics Ph.D. Qualifying Exam: Part II

August 9, 2019

Student Name: ____________________

Student UID: ____________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.

3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)

4. You can use the N(0,1) distribution table as attached.

1. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed (*iid*) Poisson(λ), where $\lambda > 0$

   (a) Find a complete sufficient statistic for λ and justify your answer.

   (b) Show that for any constant B,

   $$E\left(B^{\sum_{i=1}^{n} X_i}\right) = e^{n\lambda(B-1)}.$$

   (c) Find a UMVUE of $P(X_i = 0)$.

   (d) Find σ^2 such that

   $$\sqrt{n}\left(e^{-\bar{X}_n} - e^{-\lambda}\right) \xrightarrow{d} N(0, \sigma^2)$$

   where $\bar{X}_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$ and $\xrightarrow{d}$ indicates tends in distribution.

2. Suppose that X_1 and X_2 are independent Poisson random variables with means θ_1 and θ_2.

   (a) Find the distribution of $T = X_1 + X_2$.

   (b) Find the conditional distribution of X_1 given $T = t$.

   (c) Consider testing $H_0 : \theta_1 \leq \theta_2$ versus $H_1 : \theta_1 > \theta_2$. Find the likelihood ratio test to test H_0 versus H_1. You do not need to simplify.

3. Suppose that $X_1, X_2, \ldots, X_n$ is an iid sample from

$$f(x|\theta) = \frac{\theta}{x^2} \quad x \geq \theta$$

for $\theta > 0$.

(a) Show that there is a statistic $T = T(\mathbf{X})$ that has a monotone likelihood ratio.

(b) Find the uniformly most powerful (UMP) level α test for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

(c) Find a $100(1 - \alpha)\%$ confidence interval for θ.

4. Let Y_{ij}, $(i = 1, 2; j = 1, 2, 3$ be independent random variables such that $Y_{ij} \sim N(\mu_i, \sigma^2/i^2)$, for $j = 1, 2, 3$.

   (a) Find the least squares estimators of μ_1 and μ_2.

   (b) Find the Maximum Likelihood estimators of μ_1, μ_2, and σ^2.

   (c) Find an unbiased estimator of σ^2.

   (d) Construct a test statistic for testing $H_0 : (\mu_1, \mu_2) = (a_1\mu, a_2\mu)$ versus$H_1 : (\mu_1, \mu_2) \neq (a_1\mu, a_2\mu)$ where a_1, a_2 are known constants.

5. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from independent Exponential distributions with means μ and$\lambda\mu$ respectively.

   (a) Find the MLE of $P(Y_{(1)} > X_{(1)})$, where $X_{(1)} = \min(X_1, \ldots, X_m)$ and $Y_{(1)} = \min(Y_1, \ldots, Y_n)$.

   (b) Find jointly sufficient statistics S for (λ, μ).

6. Let X and Y be random variables such that $Y|X = x \sim \text{Poisson}(\lambda x)$, and X has density

$$f_X(x) = \frac{\theta^\theta x^{\theta-1} e^{-\theta x}}{\Gamma(\theta)}, \quad x \geq 0.$$

(a) Prove that

i. $E(Y) = \lambda$ and $Var(Y) = \lambda + \theta\lambda^2$.

ii. Y has density

$$f_Y(y;\lambda) = \frac{\Gamma(\theta + y)\lambda^y \theta^\theta}{\Gamma(\theta) y! (\theta + \lambda)^{\theta+y}}, \quad y = 0, 1, 2, \ldots$$

(b) Now suppose that $Y_1, \ldots, Y_n$ are independent random variables from the distribution given above, with Y_i having mean λ_i, and $\log(\lambda_i) = \beta z_i$, where z_i's are known covariates, $i = 1, \ldots, n.$, and assume that $\theta = 1$. Write a Fisher scoring algorithm for computing the MLE of β, and discuss its properties.

7. Let $\{X_1, \dots, X_n\}$ be a random sample from the population with density $f(x, \theta)$, where $f(x, \theta)$ is given by: $f(x, \theta) = e^{-(x-\theta)}, x > \theta; = 0$ if otherwise. Denote by $\{Y_i = i(X_{(n-i+1)} - X_{(n-i)})$ if $i = 1, \dots, n-1, Y_n = n(X_{(1)} - \theta)$, where $X_{(r)}$ is the $r-th$ order statistic, $r = 1, \dots, n$.

   (a) Show that $\{Y_i, i = 1, \dots, n\}$ are independently and identically distributed random variables with density $g(y) = e^{-y}, y > 0; = 0, y \leq 0$. Use this result to derive the UMVUE (Uniformly Minimum Varianced Unbiased estimator) of θ.

   (b) Suppose the density of the prior distribution is given by: $h(\theta) = e^{-(a-\theta)}, a > \theta; = 0, a \leq \theta$, where a is a known constant. Derive the Bayese estimator of θ and compare it with the UMVUE of θ.

8. Let $X_1, X_2, \ldots, X_n$ be be a random sample of size n from a Poisson distribution with the probability distribution function

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}, x = 0, 1, \ldots$$

(a) Find the UMVUE (Uniformly Minimum Varianced Unbiased Estimator) of $\theta e^{-\theta}$.

(b) Under either $H_0 : \theta = 25$ or $H_1 : \theta = 16$, explain why we do not require a large n to permit a reasonable normal approximation for $\sum_{i=1}^{n} X_i$.

(c) Given $H_0 : \theta = 25$ vs. $H_1 : \theta = 16$, find n to guarantee type I and type II error probabilities are less than 0.05. (i.e. $\alpha, \beta \leq 0.05$)

9. If $X_1, \ldots, X_n$ is a random sample of size n from an exponential population

$$\begin{aligned} f(x;\theta,\sigma) &= (1/\sigma)e^{-(x-\theta)/\sigma} \quad , \quad \theta \leq x < \infty, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Let $\bar{X}$ be the sample mean and $X_{(1)}$ be the smallest order statistic.

(a) Find the joint p.d.f. of $\bar{X} - X_{(1)}$ and $X_{(1)}$.

(b) Find the marginal p.d.f.s of $\bar{X} - X_{(1)}$ and $X_{(1)}$.

(c) Are $\bar{X} - X_{(1)}$ and $X_{(1)}$ independently distributed ?

(d) Find the UMVUE of θ.

10. Let $X_1, \ldots, X_n$ be a random sample of size n from $f(x;\theta) = \theta^2 x e^{-\theta x} \quad x > 0$.

(a) In testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$ the following test was used: Reject H_0 if $X_1 \leq 1$. Find the power function and size of this test.

(b) Find a most powerful size α test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

(c) Does there exist a uniformly most powerful size α test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$? If so, what is the rejection region?

(d) In testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$, among all tests based on the likelihood ratio of the form $\dfrac{L(\theta_{H_0})}{L(\theta_{H_1})}$ find a test that minimizes the sum of the sizes of the Type 1 and Type II errors based on a sample of size $n = 1$.

11. Let $X_1, \cdots, X_n$ be an i.i.d sample from the gamma density

$$f(x; \theta) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\theta x},$$

where θ is an unknown positive parameter and α is a known positive constant.

(a) Find the maximum likelihood estimator of θ.

(b) What is the limiting distribution of $\sqrt{n}(\hat{\theta} - \theta)$?

12. Let X_i's and Y_j's are independent samples from populations with densities

$$f(x, \theta_1) = (\frac{x}{\theta_1^2} e^{-x/\theta_1}),$$

$$f(y, \theta_2) = (\frac{y}{\theta_2^2} e^{-y/\theta_2}), \tag{1}$$

for $x > 0$ and $y > 0$, where $i, j = 1, 2, \cdots, n$, $\theta_1 > 0$, and $\theta_2 > 0$. Find the critical region of the likelihood ratio test of size α of $H_0 : \theta_1 = \theta_2 = \theta$ versus $H_1 : \theta_1 \neq \theta_2$.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999