Statistics Ph.D. Qualifying Exam: Part I

October 3, 2020

Student Name: ______________________________

Student UID: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Dr. Ching-Chi Yang will monitor the exam via zoom and his zoom ID is `857 148 0400` and his Passcode is `178203`

3. Write your answer right after each problem selected, attach more pages if necessary. Assemble your work in right order, scan and email your answers to `cyang3@memphis.edu`

4. You can use the N(0,1) distribution table as attached.

1. Define $Y = X_1 + ZX_2$ such that X_1, X_2, and Z are mutually independent. X_i has a Poisson distribution with mean λ_i and Z has a Bernoulli distribution with parameter p.

   (a) Find the moment generating function of Y.

   (b) Find the pdf of Y.

2. To test $H_0 : \theta = 3$ versus $H_1 : \theta = 2$ a random sample of size n is taken from

$$f(x|\theta) = \theta x^{\theta-1} \quad 0 \leq x \leq 1.$$

(a) Construct the most powerful (MP) size α test.

(b) For $n = 1$, what is the power of the test in (a)?

(c) Find a uniformly most powerful (UMP) size α test of $H_0 : \theta \leq 3$ versus $H_1 : \theta > 3$ if one exists.

3. $X_1, X_2, \ldots, X_n$ are iid from a Poisson distribution with parameter θ. Let $\tau(\theta) = P(X_i \geq 2)$.

   (a) Find an unbiased estimator of $\tau(\theta)$.

   (b) Find the UMVUE for $\tau(\theta)$

   (c) Find the Cramer-Rao lower bound for unbiased estimators of $\tau(\theta)$.

4. Let X have a uniform distribution on the interval (0, 1). Given that $X = x$, let Y have a uniform distribution on the interval $(0, 2x + 1)$.

   (a) Find $P(X > Y)$.

   (b) Find the marginal p.d.f of X, $f_1(x)$.

   (c) Find the marginal p.d.f of Y, $f_2(y)$.

   (d) Find $E(X|Y = y)$, the conditional mean of X, given that $Y = y$.

5. A random sample of size $n = 25$ is taken from the distribution with p.d.f. $f(x) = cx$, $0 < x < 2$. Let $\bar{X}$ be the sample mean of the random samples of size 25.

   (a) Find c so that $f(x)$ is indeed a p.d.f.

   (b) Find $E(\bar{X})$ and $Var(\bar{X})$.

   (c) Find, approximately, $P(1.25 < \bar{X} < 1.45)$.

   (d) Find the (exact and/or approximate) moment generating function of $\bar{X}$.

6. Let $Y_1 < Y_2 < \cdots < Y_8$ be the order statistics of eight independent observations, say $X_1, X_2, \cdots, X_8$, from a standard uniform distribution, $U(0,1)$. Let W be the number of these eight observations less than c ($0 < c < 1$ is a constant).

   (a) What is the distribution of W ?

   (b) Determine $P(Y_7 < 0.6)$.

   (c) Find $P(Y_5 < 0.6 < Y_8)$.

   (d) Find $P(0.4 < Y_6 < 0.6)$.

7. Let $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ be independent and identically distributed samples from a bivariate normal distribution with mean (μ, η) and covariance matrix $\Sigma = diag(\sigma^2, \tau^2)$, a diagonal matrix with diagonal elements σ^2 and τ^2.

   (a) Find the maximum likelihood estimate for μ, η, σ^2, and τ^2.

   (b) Find a level $(1 - \alpha)$ confidence interval for τ^2/σ^2.

   (c) If σ^2 and τ^2 are known, find a level $(1 - \alpha)$ confidence interval for $(\eta - \mu)$.

8. Suppose that $X_1, X_2, ..., X_n$ is an independent and identically distributed (iid) sample from the uniform distribution $U(-\theta, \theta)$ with an unknown parameter $\theta > 0$. Let $X_{(1)} = \min(X_1, X_2, ..., X_n)$ and $X_{(n)} = \max(X_1, X_2, ..., X_n)$ be the sample minimum and sample maximum respectively.

   (a) Prove that $T = (X_{(1)}, X_{(n)})$ is a sufficient statistic for θ.

   (b) Find $E(X_{(1)})$, $E(X_{(n)})$, and $E(X_{(n)} - X_{(1)})$.

   (c) Find various functions $g(T)$ such that $E(g(T)) = \theta$. Is T complete ? Justify your answer.

   (d) Find the maximum likelihood estimate (MLE), $\hat{\theta}$, of θ based on $X_1, X_2, ..., X_n$.

9. A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} axe^{-x/b} & \text{if} \quad x > 0, \\ 0 & \text{if otherwise.} \end{cases}$$

where a and b are two unknown positive numbers. We are also given that $E(X) = 2$. Now answer the following questions:

(a) Find the values of a and b.

(b) Find the moment generating function of X, which is defined as $M(t) = E[e^{tX}]$ for $t < 1$ here.

(c) Find the variance of X.

(d) Find $P(|X - 2| > 1)$.

Hint: $\int_0^\infty x^{k-1}e^{-x}dx = \Gamma(k) = (k-1)!$ if k is a positive integer.

10. Let $\{Y_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with mean μ and variance σ^2. Let N be a non-negative integer-valued random variable that is independent of the X_n's, and let $S_N = \sum_{i=1}^{N} X_i$, and $\bar{S}_N = \frac{S_N}{N}$.

    (a) Calculate the mean and variance of $\bar{S}_N$.

    (b) If the X_n's are I.I.D exponential random variables and N has a Geometric distribution with parameter p, that is, $P(N = n) = pq^{n-1}, \;\; n \geq 1$, prove that

$$E[N \exp(-S_N)] = \frac{p(1+\mu)}{(p+\mu)^2}.$$

11. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with mean 1 and let $Y_i = X_{(i)}$ be the ith order statistic $i = 1 \ldots, n$ with $Y_0 = 0$.

   (a) Prove that the random variables $(n+1-i)(Y_i - Y_{i-1}), i = 1, \ldots, n$ are independent and identically distributed. What is the common distribution?

   (b) Calculate $Var(Y_n - Y_1)$.

   (c) Construct a level α chi-squared test for testing $H_0 : F = F_0$ versus $H_1 : F \neq F_0$, where F is a distribution function of a continuous non-negative random variable and F_0 is the distribution function of the exponential random variable with mean 1. State the degree of freedom of your chi-squared statistic.

12. Let $Y_1, \ldots, Y_n$ be independent random variables with $Y_i \sim N(\beta x_i, \sigma^2), i = 1 \ldots n$, with β and σ^2 are unknown. You wish to test $H_0 : \beta = 0$ versus $H_1 : \beta \neq 0$

   (a) Construct a likelihood ratio test for a test of size $\alpha = 0.05$. Write down the rejection region explicitly.

   (b) Now consider testing $H_0 : \beta \leq 0$ versus $H_1 : \beta > 0$

   (c) Show that the distribution has monotone likelihood ratio property in the sufficient statistic and hence write down an explicit form of the UMP test in terms of tabulated percentiles.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999