Statistics Ph.D. Qualifying Exam: Part II

October 9, 2020

Student Name: ______________________________

Student UID: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Dr. Ching-Chi Yang will monitor the exam via zoom and his zoom ID is `857 148 0400` and his Passcode is `178203`

3. Write your answer right after each problem selected, attach more pages if necessary. Assemble your work in right order, scan and email your answers to `cyang3@memphis.edu`

4. You can use the N(0,1) distribution table as attached.

1. X_1, X_2 is a random sample from

$$f(x) = \begin{cases} 1.5x^2 & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the cdf and pdf of $Y = X_1^2$.

(b) Let $X_{(2)}$ be the maximum order statistic. Find the pdf of $X_{(2)}$.

(c) Find $E(X_1 X_2)$ and $Var(X_1 X_2)$.

2. Let X be a binomial random variable with parameters n and p. For an event that occurs with probability θ, the odds of the event occurring are $\dfrac{\theta}{1-\theta}$.

   (a) Show that
$$E\left(\frac{X/n}{1-X/n}\right) \geq \frac{p}{1-p}$$

   (b) Use the delta method to find expressions for $E\left(\dfrac{X/n}{1-X/n}\right)$ and $var\left(\dfrac{X/n}{1-X/n}\right)$.

   (c) Suppose drivers who don't wear seatbelts are 3 times as likely to die as those who do wear a seatbelt. Suppose the odds of a randomly selected driver wearing a seat belt are 9 to 1 in favor of wearing a seatbelt. What are the odds that a driver was wearing a seatbelt given the driver died in a car crash?

3. Let $U_1 \sim U(0,1)$ and $U_2 \sim U(0,1)$ where U_1 and U_2 are independent. Let

$$X_1 = \sqrt{-2\,\log(U_1)}\,\cos(2\pi U_2) \text{ and } X_2 = \frac{1}{2\pi}\sqrt{-2\,\log(U_1)}\,\sin(2\pi U_2).$$

   (a) Find the joint density function of (X_1, X_2).
   (b) Find the marginal density function of X_1 and X_2.
   (c) Find the distribution of $X_1^2 + X_2^2$ which is $-2\log(U_1)$.
   (d) Find the distribution of X_2/X_1 which is $\tan(2\pi U_2)$.

   Hint: $\frac{d}{dx} tan^{-1}(x) = (1 + x^2)^{-1}$

4. Let X be a single observation from a distribution with density function

$$f_\theta(x) = \begin{cases} \frac{x}{\theta} & \text{if } 0 \le x < \theta, \\ \frac{2-x}{2-\theta} & \text{if } \theta \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

for an unknown parameter $\theta \in \Theta = (0, 2)$.

(a) Find the rejection region for the most powerful level $\alpha = 0.05$ of $H_0 : \theta = 1$ vs $H_a : \theta = 0.1$. Please clearly specify the critical value.

(b) What is the ratio of likelihoods under $H_0 : \theta = 1$ vs $H_a : \theta = \theta_1$?

(c) Is there a uniformly most powerful level $\alpha = 0.05$ test of $\alpha = 0.05$ of $H_0 : \theta \ge 1$ vs $H_a : \theta < 1$?

5. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with density $f(x, \theta)$, where $f(x, \theta)$ is given by: $f(x, \theta) = e^{-(x-\theta)}, x > \theta; = 0$ if otherwise. Denote by $\{Y_i = i(X_{(n-i+1)} - X_{(n-i)})$ if $i = 1, \ldots, n-1, Y_n = n(X_{(1)} - \theta)$, where $X_{(r)}$ is the $r-th$ order statistic, $r = 1, \ldots, n$.

   (a) Show that $\{Y_i, i = 1, \ldots, n\}$ are independently and identically distributed random variables with density $g(y) = e^{-y}, y > 0; = 0, y \leq 0$. Use this result to derive the UMVUE (Uniformly Minimum Varianced Unbiased estimator) of θ.

   (b) Suppose the density of the prior distribution is given by: $h(\theta) = e^{-(a-\theta)}, a > \theta; = 0, a \leq \theta$, where a is a known constant. Derive the Bayese estimator of θ and compare it with the UMVUE of θ.

6. Let $\{X_1, \ldots, X_n\}$ be a random sample from a population with density $f(x, \theta) = \{\theta^2 + 2\theta\,(1-\theta)\}^x(1-\theta)^{2(1-x)}, x = 0, 1, 0 < \theta < 1$.

   (a) Derive the moment estimator of θ.

   (b) Explain how to derive the MLE of θ by using the EM-algorithm.

   (c) Which estimator you would prefer for estimating θ?

7. Suppose that an random sample $X_1, ..., X_n$ taken from an exponential distribution with mean θ and another random sample $Y_1, ..., Y_m$ taken from exponential distribution with mean μ.

   (a) Derive the likelihood ratio test for $H_0 : \theta = \mu$ vs. $H_1 : \theta \neq \mu$.

   (b) Show the test above can be based on the statistics

$$T = \frac{\sum X_i}{\sum Y_i}.$$

   (c) Find the sampling distribution of T when H_0 is true.

8. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from Normal(μ_1, σ^2) and Normal (μ_2, σ^2) populations respectively, where σ^2 is unknown.

   (a) Construct a likelihood ratio test of

   $$H_0 : \mu_1 = \mu_2 \quad \text{versus} \quad H_1 : \mu_1 \neq \mu_2.$$

   with level of significance α.

   (b) Give the critical values of this test in terms of percentiles of one of the standard distributions.

   (c) Is the likelihood ratio test, uniformly most powerful? Why or Why not?

9. Let $X_1, X_2, \ldots X_n$ be a independent Normal(μ, σ^2) random variables.

   (a) Prove that $\bar{X}$ and $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2$ are independent.

   (b) Prove that S^2/σ^2 has a chi-squared distribution.

   (c) Let $g(x)$ be a continuous function of x. Find C such that $C(g(\bar{X}) - g(\mu))^2/S^2$ has an F distribution. What are the degrees of freedom?

   (d) For testing $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$, explain fully how you would calculate the power of a size α test at $\mu = 1$. What distrubution would you use?

10. Suppose that $X|n, \theta$ has a binomial distribution with parameter θ. Suppose we put independent prior distributions on n and θ, with n having Poisson(λ) prior and θ having a Beta(α, β) prior, where α and β are known hyperparameters.

   (a) Prove that the posterior density of θ given $X = x$ and n is Beta($x + \alpha, n - x + \beta$).

   (b) Prove that the posterior probability function of $n + X$ given $X = x$ and θ is Poisson$[(1 - \theta)\lambda]$.

   (c) Suppose $\alpha = \beta = 1$ and $X = 10$, explain in details how you can obtained 100 samples of n's from the **posterior distribution** of n given $X = 10$.

11. Suppose that the discrete random variable Y has the negative binomial distribution

$$P_Y(y) = \binom{y+k-1}{k-1}\pi^k(1-\pi)^y, y = 0, 1, \cdots, \infty, 0 < \pi < 1, \tag{1}$$

with k a known positive integer.

(a) Derive an explicit expression for $E[Y!/(Y-r)!]$ where r is a nonnegative integer.

(b) Then, use this result to find $E(X)$ and $V(X)$ when $X = (Y + k)$.

12. Two balls are selected sequentially at random without replacement from an urn containing $N(> 1)$ balls numbered individually from 1 to N. Let the discrete random variable X be the number on the first ball selected, and let the discrete random variable Y be the number on the second ball selected.

   (a) Provide an explicit expression for the joint distribution of the random variables X and Y, and also provide explicit expressions for the marginal distributions of X and Y.

   (b) Provide an explicit expression for $P(X \geq (N-1)|Y = y)$, where y is a fixed positive integer satisfying the inequality $1 \leq y \leq N$.

   (c) Derive an explicit expression for $corr(X, Y)$, the correlation between X and Y. Find the limiting value of $corr(X, Y)$ as $N \rightarrow \infty$, and then comment on your finding.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999