Statistics Ph.D. Qualifying Exam: Part II

October 8, 2021

Student Name: ____________________

Student UID: ____________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. **Do not** write your answers on the back.
3. Assemble your work in right order and in the original problem order. (Including the ones that you do not select)
4. You can use the N(0,1) distribution table as attached.

1. Consider a sequence of Bernoulli trials with probability of success ("S") is p. Let Y be the number of failures ("F") needed until k "S" and the distribution of Y will follow a negative binomial distribution, $Y \sim NB(k, p)$.

   (a) Find $P(Y = y)$.

   (b) Derive $\mu = E(Y)$ and $\sigma^2 = Var(Y)$.

   (c) Let $p = k/(k + \mu)$ and show that

   $$P(Y = y) = f(y; k, \mu) = \frac{\Gamma(y + k)}{\Gamma(k)\Gamma(y + 1)} \left(\frac{k}{\mu + k}\right)^k \left(1 - \frac{k}{\mu + k}\right)^y, \quad y = 0, 1, 2, ...$$

   Note that k does not have to be an integer under this formulation.

   (d) Show that as $k \to \infty$, $Var(Y) \to \mu$ and Y is approximately a Poisson random variable.

2. Consider independent random samples $X_{i1}, \ldots, X_{in}$ $(i = 1, 2)$ from the uniform populations:

$$f(x_i) = \frac{1}{\theta_i}, \quad 0 < x_i < \theta_i, \quad \theta_i > 0, \quad i = 1, 2.$$

Let $Y_i = \max(X_{i1}, \ldots, X_{in})$, $i = 1, 2$, and $Y = \max(Y_1, Y_2)$.

(a) Show that the likelihood ratio statistic to test the hypothesis $H_0 : \theta_1 = \theta_2$ is given by

$$Z = \left(Y_1 Y_2 / Y^2\right)^n.$$

(b) Obtain the exact distribution of $-2 \log Z$ under H_0.

3. Assume that random variables X and Y follow a trinomial distribution with the probability distribution function

$$P(X = x, Y = y|\theta, \lambda) = \frac{n!}{x!y!(n-x-y)!}\theta^x \lambda^y (1-\theta-\lambda)^{n-x-y}, \quad x \geq 0, y \geq 0, x+y \leq n.$$

Suppose further that (θ, λ) have a uniform prior distribution over the triangular region $0 < \theta + \lambda < 1$.

(a) Find the marginal distribution of X.

(b) Find the marginal distribution of $X + Y$.

(c) Find the posterior distribution of (θ, λ) given X and Y.

(d) Find the Bayes estimator of θ under square error loss function.

4. Suppose that X_1, ..., X_n is a random sample with common beta density of the form $f(x|\theta) = \theta(\theta+1)x^{\theta-1}(1-x)$ on $0 \le x \le 1$ (0 elsewhere), $\theta > 0$.

   (a) Find the method of moments estimator for θ.

   (b) Show that $T = \Pi_{i=1}^{n} X_i$ is a minimal sufficient statistic for θ.

5. Let (X_1, Y_1), ..., (X_n, Y_n) be independent and identically distributed sample from a bivariate normal distribution with mean (μ, η) and covariance matrix $\Sigma = diag(\sigma^2, \tau^2)$.

   (a) If $\mu, \eta, \sigma^2, \tau^2$ are unknown parameters, find the sufficient and complete statistic for $(\mu, \eta, \sigma^2, \tau^2)$.

   (b) Find the maximum likelihood estimate and uniformly minimum variance unbiased estimate (UMVUE) for μ, η, σ^2, and τ^2

6. Suppose (X_1, Y_1), ..., (X_n, Y_n) are random samples from a bivariate distribution. The conditional distribution of X_i given $Y_i = y_i$ is $N(y_i, 1)$ and the distribution for Y_i is $N(0, e^\theta)$.

   (a) Find the CDF for

$$\max_{i=1,\ldots,n} \{X_i + Y_i\}.$$

   (b) Show that $\sum_{i=1}^n Y_i^2$ is a complete and sufficient statistic for θ.

   (c) Find the uniformly minimum variance unbiased estimator (UMVUE) for e^θ.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x|\theta)$ where

$$f(x|\theta) = \sqrt{\frac{2}{\pi}} \frac{x^2}{\theta^3} \exp\left(\frac{-x^2}{2\theta^2}\right) \qquad x > 0, \theta > 0$$

For this distribution $E(X) = 2\theta\sqrt{\frac{2}{\pi}}$ and $Var(X) = \dfrac{\theta^2(3\pi - 8)}{\pi}$

(a) Find a minimal sufficient statistic for θ.

(b) Find a method of moments estimate for θ^2.

(c) Find a maximum likelihood estimator for θ^2.

(d) Find the Fisher information for $\tau = \theta^2$ in the sample of n observations.

8. Let X be a normal random variable with mean 0 and variance 1. Let Y be uniform(0,1) where X and Y are independent. Let $U = X + Y$ and $V = X - Y$.

   (a) Find the joint distribution of U and V.

   (b) Find the marginal distribution of U.

   (c) Find the cov(U,V). Are U and V independent ? Explain.

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from distribution

$$f(x|\theta) = \frac{\theta}{10}\left(\frac{x}{10}\right)^{\theta-1} \qquad 0 < x < 10$$

   (a) Find the form of the most powerful level α test for testing

$$H_0 : \theta := \theta_0 \ \ vs \ \ H_1 : \theta = \theta_1$$

   (b) Find the form of the UMP (uniformly most powerful) test for

$$H_0 : \theta \leq \theta_0 \ \ vs \ \ H_1 : \theta > \theta_1$$

   for $\theta_1 > \theta_0 > 0$.

10. Let $X_1, \ldots, X_n$ be a random sample from a Normal $(\mu, 1/\tau)$ population. Assume the following prior specifications on μ and τ: $\mu|\tau \sim N(\mu_0, \frac{1}{\lambda\tau})$, $\tau \sim Gamma(\alpha, \beta)$.

   (a) Show that this prior specification is conjugate for this problem.

   (b) Find the posterior distribution of $\sqrt{\frac{\lambda\alpha}{\beta}}(\mu - \mu_0)$.

11. Let $Y_1, \ldots, Y_N$ be independent random variable such that $Y_i \sim \text{Binomial}(n_i, p_i)$, where

$$P_i = \frac{1}{1 + e^{\alpha + \beta x_i}},$$

and x_i is a fixed covariate, $i = 1, \ldots, N$.

(a) Find a set of jointly sufficient statistics for (α, β).

(b) Suppose $\tilde{\alpha}, \tilde{\beta}$ are the estimates of α and β that minimize

$$Q = \sum_{i=1}^{N} n_i \hat{p}_i (1 - \hat{p}_i)(\alpha + \beta x_i - l(Y_i/n_i))^2,$$

where $\hat{p}_i = Y_i/n_i$ and $l(Y_i/n_i) = log(\frac{Y_i/n_i}{1-Y_i/n_i}), i = 1, \ldots, N$. Find $\tilde{\alpha}$ and $\tilde{\beta}$.

(c) Let $\hat{\alpha}, \hat{\beta}$ be MLE's of α and β. Give an argument to show that

$$\text{Mean Square Error}(\hat{\alpha} + \hat{\beta}) \leq \text{Mean Square Error}(\tilde{\alpha} + \tilde{\beta})$$

12. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from Exponential(λ) and Exponential(μ) populations respectively.

    (a) Construct a likelihood ratio test of

$$H_0 : \lambda = \mu \quad \text{versus} \quad H_1 : \lambda \neq \mu.$$

    (b) Give the critical values of this test in terms of percentiles of one of the standard distributions.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999