Statistics Ph.D. Qualifying Exam: Part I

October 7, 2022

Student Name: ______________________

Student UID: ______________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. Assemble your work in right order.
3. You can use the N(0,1) distribution table as attached.

1. Let Y be the sum of the observations of a random sample from a Poisson distribution with mean θ. Let the prior pdf of θ be gamma with parameters α and β,

$$f(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\theta/\beta} \quad \alpha, \beta > 0.$$

   (a) Find the posterior pdf of θ, given that $Y = y$.

   (b) If the loss function is $[w(y) - \theta]^2$, find the point estimate $w(y)$ which minimize the loss function.

2. Let Y_1, Y_2, ..., Y_n be i.i.d random variable from a distribution with probability density function,

$$f(y) = \frac{\theta^2}{1+\theta}\,(1+y)\,e^{-\theta y}$$

where $y > 0$ and $\theta > 0$.

(a) Find the maximum likelihood estimator (MLE) of θ.

(b) Find the MLE of $1/\theta$.

3. Let X_1, X_2, ..., X_n be i.i.d random variable from a distribution with probability density function,

$$f(x|\theta) = \frac{2x}{\theta} e^{-x^2/\theta}$$

where $x > 0$.

(a) Show that X_1^2 is an unbiased estimator of θ.

(b) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of θ.

(c) Find the uniformly minimum variance unbiased estimator (UMVUE) of θ.

4. Let X and Y be independent random variables with $X \sim$ Exponential(λ) and $Y \sim$ Exponential(μ). Let Z and W be two random variables where $Z = \min(X, Y)$ and

$$W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases}$$

(a) Find the joint distribution of Z and W.

(b) Prove that Z and W are independent. (Hint: Show that $P(Z \leq z | W = i) = P(Z \leq z)$ for $i = 0$ or 1.)

5. Consider the following model: (X_i, p_i) are independent pairs with

$$X_i | p_i \sim \text{Bernouilli}(p_i),$$
$$p_i \sim \text{Beta}(\alpha, \beta), \quad i = 1, ..., n$$

Let $Y = X_1 + X_2 + \ldots + X_n$.

(a) Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

(b) Find the marginal pdfs of X_i's, $i = 1, ..., n$, and use them to deduce the pdf of Y.

6. Let $X_1, ..., X_n$ be a random sample from the inverse Gaussian pdf

$$f(x|\mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(-\frac{(\lambda - \mu)^2}{2\mu^2 x}\right), \quad x > 0$$

(a) Find the method of moments estimator of μ and λ. (Hint: $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \frac{\mu^3}{\lambda}$)

(b) Find the MLE of μ and λ.

(c) Assume that λ is known. Is $\hat{\mu}$, the MLE for μ, unbiased? Does the variance of $\hat{\mu}$ attain the Rao-Cramer lower bound? Justify your answers.

7. Let U_1, U_2, U_3 be iid Bernoulli with parameter π, that is

$$P(U_i = u_i) = \pi^{u_i}(1-\pi)^{1-u_i} \quad u_i = 0, 1 \;\; i = 1, 2, 3 \;\; 0 < \pi < 1$$

Let W be independent of U_i, $i = 1, 2, 3$ with a Bernoulli distribution with parameter θ, that is

$$P(W = w) = \theta^{w}(1-\theta)^{1-w} \quad w = 0, 1 \quad 0 < \theta < 1.$$

Let $X = WU_1 + (1-W)U_2$ and $Y = WU_1 + (1-W)U_3$.

(a) Find the correlation between X and Y.

(b) Find $E(Y|X = x)$.

8. $X_1, X_2, \ldots, X_n$ are iid with pdf given by

$$P(X = x; \theta) = (1 - \theta)\theta^{x-1} \quad x = 1, 2, \ldots \quad 0 < \theta < 1.$$

Find the uniformly most powerful (UMP) test of size α for testing $H_0 : \theta = 1/3$ versus the alternative hypothesis $H_1 : \theta > 1/3$.

9. $X_1, X_2, \ldots, X_n$ is a random sample from density

$$f_X(s;\theta) = \theta e^{-\theta x} \quad x > 0 \quad \theta > 0.$$

(a) Find the UMVUE of θ.

(b) Find the Cramer Rao lower bound of any unbiased estimator of θ.

(c) Find the mean squared error of the UMVUE.

10. Let X_1 and X_2 be two independent random variables, each with p.d.f $f(x) = 1/2, 0 < x < 2$. Therefore, the joint p.d.f. of X_1 and X_2 is

$$f(x_1, x_2) = 1/4, \quad 0 < x_1 < 2, 0 < x_2 < 2.$$

Let us consider

$$Y_1 = \frac{X_1}{X_1 + X_2}, \quad Y_2 = X_1 + X_2.$$

(a) Find the joint pdf of Y_1 and Y_2. $g(x, y)$.

(b) Find the marginal p.d.f of Y_1, $g_1(y)$.

(c) Find the marginal p.d.f of Y_2, $g_2(y)$. .

11. Let $Y_1 < Y_2 < \cdots < Y_{10}$ be the order statistics of ten independent observations, say $X_1, X_2, \cdots, X_{10}$, from a standard uniform distribution, $U(0,1)$. Let W be the number of these ten observations less than c ($0 < c < 1$ is a constant).

(a) What is the distribution of W ?

(b) What is the p.d.f. of Y_3 ? What is its distribution ?

(c) Determine $P(Y_3 < 0.6)$.

(d) Find $P(Y_5 < 0.6 < Y_8)$.

(e) Find $P(0.4 < Y_6 < 0.6)$.

12. Let X_1, X_2 be two independent random variables taking values from $0, 1, 2, \cdots, 9$ with equal probability.

   (a) Define the mgf of X_1.

   (b) Define the mgf of $10X_2$.

   (c) Let $W = 10X_2 + X_1$. Find the mgf of W

   (d) Find the pmf of W.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999