Statistics Ph.D. Qualifying Exam: Part II

October 14, 2022

Student Name: ____________________

Student UID: ____________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. Assemble your work in right order.

3. You can use the N(0,1) distribution table as attached.

1. Let $X_1, \ldots, X_n$ be i.i.d. having the uniform distribution on the interval (θ_1, θ_2). Find the uniformly minimum variance unbiased estimator (UMVUE) for

   (a) θ_1.

   (b) θ_2.

   (c) $\theta_2 - \theta_1$.

2. Let $\{X_1, \ldots, X_n\}$ be a random sample from the population with density $f(x, \theta) = \theta\, x^{\theta-1}, 0 < x < 1, \theta > 0$.

   (a) Derive the UMP (Uniformly Most Powerful) size α test for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

   (b) What is the power function of the above UMP test ?

   (c) Is there an UMP size-α test for testing $H_0 : \theta = 1$ versus $H_2 : \theta \neq 1$? Justify your answer.

3. Let $X_1, \ldots, X_n$ be a random sample from a distribution with density $f(x|\theta)$, where $f(x|\theta)$ is given by

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & , \quad \theta > 0, 0 < x < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Assume that the prior distribution of θ is

$$p(\theta) = \frac{\lambda^m}{\Gamma(m)} \theta^{m-1} e^{-\lambda\theta}, \lambda > 0, m > 1$$

where λ and m are known constants.

(a) Identify the posterior distribution of θ given the observed data $x_1, \ldots, x_n$.

(b) Obtain the Bayes estimator $\hat{\theta}$ of θ when the loss function of θ is $l = c(\theta - \hat{\theta})^2$ where $c > 0$.

(c) Explain how to obtain a 95% Bayesian interval for θ.

4. Let x_1, ..., x_n be nonzero constants with $\sum_{i=1}^n (x_i - \bar{x})^2 > 0$. Suppose we observe (x_i, Y_i), $i = 1$, ..., n from the following regression model:

$$Y_i = \beta x_i + \epsilon_i$$

where ϵ_1, ..., ϵ_n are a random sample from the Gaussian distribution $N(0, \sigma^2)$.

(a) Find the mean and variance of

$$\frac{\sum_{i=1}^n x_i Y_i}{\sum_{i=1}^n x_i^2}.$$

(b) Find the mean and variance of

$$\frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n x_i}.$$

(c) Find the estimator of β that has the smallest variance among unbiased estimators of β of the form

$$\sum_{i=1}^n c_i Y_i.$$

(d) Now suppose $\epsilon_i \sim N(0, x_i^2)$ and ϵ_1, ..., ϵ_n are independent. Find the estimator of β that has the smallest variance among unbiased estimators of β of the form

$$\sum_{i=1}^n c_i Y_i.$$

5. Suppose $Y_1, ..., Y_n$ are independent binomial (m_i, ρ) where m_i are fixed positive integers.

(a) Let

$$T_1 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n m_i}.$$

Find the mean and the variance of T_1. Is T_1 an unbiased estimator of ρ?

(b) Let

$$T_2 = \frac{1}{n}\sum_{i=1}^n \frac{Y_i}{m_i}.$$

Find the mean and the variance of T_2. Is T_2 an unbiased estimator of ρ?

(c) Which estimator is better to estimate ρ? Justify your answer.

6. Let Y_1, ..., Y_n are i.i.d. random variables from a distribution with pdf

$$f(y) = \frac{\theta}{2(1+|y|)^{\theta+1}}$$

where $\theta > 0$ and $y \in \mathbb{R}$.

(a) Find a complete sufficient statistic.

(b) Find the Fisher information.

(c) Find the uniformly minimum variance unbiased estimator (UMVUE) for θ.

7. Suppose $X_i \sim N(\mu, \sigma^2)$, $i = 1, \cdots, n$ and $Z_j \sim N(0,1)$, $j = 1, \cdots, k$ and all variables are independent. Find the distribution of each of the following variables.

   (a) $X_2 + 2X_3$.

   (b) $\frac{\bar{X}}{\bar{Z}}$.

   (c) $\frac{\bar{X}}{\sigma^2} + \bar{Z}$.

8. Consider independent random samples from two exponential distributions, $X_i \sim EXP(\theta_1)$ and $Y_j \sim EXP(\theta_2)$; $i = 1, \cdots, n_1$, $j = 1, \cdots, n_2$.

   (a) Show that $(\theta_2/\theta_1)(\bar{X}/\bar{Y}) \sim F(2n_1, 2n_2)$.

   (b) Derive a $100\gamma\%$ confidence interval for θ_2/θ_1.

9. Let

$$Y_i = \beta x_i + \epsilon_i,$$

where x_i is a fixed covariate and $\epsilon_i \sim N(0, \sigma_0^2), i = 1, \ldots, n$ where σ_0^2 is known and β is an unknown parameter.

(a) Find the maximum likelihood estimator of β.

(b) Using a $N(0, 1)$ as a prior distribution for β, find the posterior distribution of β.

(c) How does this posterior distribution relate to the MLE for β?

10. Let X_1, X_2, X_3, and X_4 be i.i.d. exponential random variables with the pdf $f(x) = \theta \exp(-\theta x),\ x, \theta > 0$. Find the pdf of

   (a) $Y_1 = \min(X_1, X_2, X_3, X_4)$.

   (b) $Y_4 = \max(X_1, X_2, X_3, X_4)$.

   (c) Random vector (Y_1, Y_4).

   (d) $Y = Y_4 - Y_1 = \max(X_1, X_2, X_3, X_4) - \min(X_1, X_2, X_3, X_4)$.

11. Consider an i.i.d. sample $X_1, \ldots, X_n$ from the Pareto distribution with the pdf

$$p(x|b) = \frac{b}{x^{b+1}}, \quad 1 < x < \infty$$

(a) Show that $Y_i = \ln(X_i)$, $i = 1, \ldots, n$, have exponential distributions and use this information to find the distribution of the sufficient statistic.

(b) Construct the Likelihood Ratio Test (LRT) for $H_0 : b = b_0$ versus $H_1 : b > b_0$.

12. Let $\mathbf{X} = (X_1, ..., X_n)$ be a random sample from the Poisson distribution with mean λ. Let the prior distribution for λ be a Gamma(a, b) distribution

$$p(x|a,b) = \frac{x^{a-1}e^{-x/b}}{b^a\Gamma(a)}, \quad x > 0$$

where $a = 4$ and $b = 2$.

(a) Find the posterior distribution of λ given data $\mathbf{X}$.

(b) Construct a Bayesian test for $H_0 : \lambda \geq 2$ versus $H_1 : \lambda < 2$. What is the rejection region of this test? Assume that H_0 and H_1 are equally important.

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999