# Statistics Ph.D. Qualifying Exam: Part I

October 6, 2023

Student Name: ____________________

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1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Selected | | | | | | | | | | | | |
| Scores | | | | | | | | | | | | |

2. Write your answer right after each problem selected, attach more pages if necessary. Assemble your work in right order.

3. You can use the N(0,1) distribution table as attached.

1. For the hierarchical model

$$Y|\Lambda \sim \text{Poisson}(\Lambda) \quad \text{and} \quad \Lambda \sim \text{gamma}(\alpha, \beta),$$

(a) Find the marginal distribution of $Y$

(b) Find the mean and variance of $Y$

(c) Show that the marginal distribution of $Y$ is a negative binomial if $\alpha$ is an integer.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a distribution whose pdf is

$$f(x;\theta) = \frac{1}{\theta} x^{(1-\theta)/\theta}, \quad 0 < x < 1,\, 0 < \theta < \infty.$$

(a) Find the maximum likelihood estimator of $\theta$.

(b) Is your estimator in part (a) an unbiased estimator of $\theta$? Justify your answer.

(c) Find a complete sufficient statistic for $\theta$.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a Poisson distribution with mean $\lambda$.

   (a) Find a uniformly most powerful (UMP) test of $H_0 : \lambda \leq \lambda_0$ versus $H_1 : \lambda > \lambda_0$.

   (b) Consider the specific case $H_0 : \lambda \leq 1$ versus $H_1 : \lambda > 1$. Use the Central Limit Theorem to determine the sample size $n$ so a UMP test satisfies $P(\text{reject } H_0 | \lambda = 1) = 0.05$ and $P(\text{reject } H_0 | \lambda = 2) = 0.9$.

4. Let $X_1$, $X_2$, ..., $X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \frac{1}{2\theta}, \quad -\theta < x < \theta, \quad \theta > 0.$$

(a) Find a sufficient statistics for $\theta$.

(b) Find the maximum likelihood estimator of $\theta$.

(c) Find a best unbiased estimator of $\theta$.

5. Let $X_1$, $X_2$, ..., $X_n$ be i.i.d. exponential($\lambda$), that is, $f(x) = \frac{e^{-x/\lambda}}{\lambda}$ for $x \geq 0$ and $\lambda > 0$.

   (a) Find an unbiased estimator of $\lambda$ based only on $Y = min\{X_1, X_2, ..., X_n\}$.

   (b) Find a better estimator than the one in part (a). Prove that it is better.

6. Let $X$, $Y$, and $Z$ are independent random variables.

   (a) Assume $X, Y$, and $Z$ are uniformly distributed over $(0, 1)$. Find $P(X < Z$ and $Y < Z)$.

   (b) Assume $X$, $Y$, and $Z$ are normally distributed, each with mean 1. Do you have enough information to compute $P(2X - 4Y > 3Z - 5X)$? If your answer is yes, find the value, otherwise explain what is missing.

7. Let $X_1, \ldots, X_n$ be a independent random variables such that $X_i \sim Normal(\theta a_i, \sigma^2)$, where $\sum_{i=1}^{n} a_i = 1$.

   (a) Find the maximum likelihood estimators of $\theta$ and $\sigma^2$.

   (b) When n=2, use the MLE of $\sigma^2$ to construct an unbiased estimator of $\sigma^2$.

8. Let $X_1, \ldots, X_n$ be a random sample an exponential population with parameter $\theta, n > 2$. That is each $X$ has density

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0$$

Suppose we put a Gamma $(\alpha, \beta)$ prior on $\theta$.

(a) Show that this prior is conjugate.

(b) Find the Bayes estimator of $\theta$ if we use the loss function $L(\theta, a) = (\theta - a)^2/\theta$.

9. Let $X_1, \ldots, X_n$ be a random sample from a population with density

$$f(x;\theta) = \theta x e^{-\frac{\theta x^2}{2}}, \qquad x > 0.$$

(a) Find a sufficient statistic for $\theta$.

(b) Construct a uniformly most powerful level $\alpha$ test for testing $H_0 : \theta = 1$ versus $H_1 : \theta > 1$.

(c) If $n = 400$, find approximately the power of the test at $\theta = 2$, when $\alpha = 0.05$

10. Let $X$ and $Y$ be two independent random variables with $U(0,1)$ distribution. Define $U = X + Y, V = X/(X+Y)$.

   (a) Find the joint density of $U$ and $V$.

   (b) Find the marginal density of $U$.

   (c) Find the marginal density of $V$.

11. Let $X_1, \ldots, X_n$ be a random sample of size $n$ from a population with density

$$f(x|\theta) = \theta(1+x)^{-(\theta+1)}, \quad x > 0, \theta > 0.$$

Suppose that $\theta$ has a prior distribution which is exponential with a known mean $\mu$.

(a) Find the posterior distribution of $\theta$.

(b) Find the Bayes estimator of $\theta$ under the squared error loss function.

12. Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. with $U(0,1)$ distribution. Let $Y_{(j)}$ be the $j$th order statistics ($r$-th smallest) of $Y_1, Y_2, \ldots, Y_n$.

   (a) Derive the distribution of $Y_{(1)}$.

   (b) Derive the distribution of $Y_{(n)}$.

   (c) Derive the distribution of $Y_{(n)} - Y_{(1)}$.

Table of P(Z < z), Z ~N(0,1)

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.50000 | 0.50399 | 0.50798 | 0.51197 | 0.51595 | 0.51994 | 0.52392 | 0.52790 | 0.53188 | 0.53586 |
| **0.1** | 0.53983 | 0.54380 | 0.54776 | 0.55172 | 0.55567 | 0.55962 | 0.56356 | 0.56749 | 0.57142 | 0.57535 |
| **0.2** | 0.57926 | 0.58317 | 0.58706 | 0.59095 | 0.59483 | 0.59871 | 0.60257 | 0.60642 | 0.61026 | 0.61409 |
| **0.3** | 0.61791 | 0.62172 | 0.62552 | 0.62930 | 0.63307 | 0.63683 | 0.64058 | 0.64431 | 0.64803 | 0.65173 |
| **0.4** | 0.65542 | 0.65910 | 0.66276 | 0.66640 | 0.67003 | 0.67364 | 0.67724 | 0.68082 | 0.68439 | 0.68793 |
| **0.5** | 0.69146 | 0.69497 | 0.69847 | 0.70194 | 0.70540 | 0.70884 | 0.71226 | 0.71566 | 0.71904 | 0.72240 |
| **0.6** | 0.72575 | 0.72907 | 0.73237 | 0.73565 | 0.73891 | 0.74215 | 0.74537 | 0.74857 | 0.75175 | 0.75490 |
| **0.7** | 0.75804 | 0.76115 | 0.76424 | 0.76730 | 0.77035 | 0.77337 | 0.77637 | 0.77935 | 0.78230 | 0.78524 |
| **0.8** | 0.78814 | 0.79103 | 0.79389 | 0.79673 | 0.79955 | 0.80234 | 0.80511 | 0.80785 | 0.81057 | 0.81327 |
| **0.9** | 0.81594 | 0.81859 | 0.82121 | 0.82381 | 0.82639 | 0.82894 | 0.83147 | 0.83398 | 0.83646 | 0.83891 |
| **1.0** | 0.84134 | 0.84375 | 0.84614 | 0.84849 | 0.85083 | 0.85314 | 0.85543 | 0.85769 | 0.85993 | 0.86214 |
| **1.1** | 0.86433 | 0.86650 | 0.86864 | 0.87076 | 0.87286 | 0.87493 | 0.87698 | 0.87900 | 0.88100 | 0.88298 |
| **1.2** | 0.88493 | 0.88686 | 0.88877 | 0.89065 | 0.89251 | 0.89435 | 0.89617 | 0.89796 | 0.89973 | 0.90147 |
| **1.3** | 0.90320 | 0.90490 | 0.90658 | 0.90824 | 0.90988 | 0.91149 | 0.91309 | 0.91466 | 0.91621 | 0.91774 |
| **1.4** | 0.91924 | 0.92073 | 0.92220 | 0.92364 | 0.92507 | 0.92647 | 0.92785 | 0.92922 | 0.93056 | 0.93189 |
| **1.5** | 0.93319 | 0.93448 | 0.93574 | 0.93699 | 0.93822 | 0.93943 | 0.94062 | 0.94179 | 0.94295 | 0.94408 |
| **1.6** | 0.94520 | 0.94630 | 0.94738 | 0.94845 | 0.94950 | 0.95053 | 0.95154 | 0.95254 | 0.95352 | 0.95449 |
| **1.7** | 0.95543 | 0.95637 | 0.95728 | 0.95818 | 0.95907 | 0.95994 | 0.96080 | 0.96164 | 0.96246 | 0.96327 |
| **1.8** | 0.96407 | 0.96485 | 0.96562 | 0.96638 | 0.96712 | 0.96784 | 0.96856 | 0.96926 | 0.96995 | 0.97062 |
| **1.9** | 0.97128 | 0.97193 | 0.97257 | 0.97320 | 0.97381 | 0.97441 | 0.97500 | 0.97558 | 0.97615 | 0.97670 |
| **2.0** | 0.97725 | 0.97778 | 0.97831 | 0.97882 | 0.97932 | 0.97982 | 0.98030 | 0.98077 | 0.98124 | 0.98169 |
| **2.1** | 0.98214 | 0.98257 | 0.98300 | 0.98341 | 0.98382 | 0.98422 | 0.98461 | 0.98500 | 0.98537 | 0.98574 |
| **2.2** | 0.98610 | 0.98645 | 0.98679 | 0.98713 | 0.98745 | 0.98778 | 0.98809 | 0.98840 | 0.98870 | 0.98899 |
| **2.3** | 0.98928 | 0.98956 | 0.98983 | 0.99010 | 0.99036 | 0.99061 | 0.99086 | 0.99111 | 0.99134 | 0.99158 |
| **2.4** | 0.99180 | 0.99202 | 0.99224 | 0.99245 | 0.99266 | 0.99286 | 0.99305 | 0.99324 | 0.99343 | 0.99361 |
| **2.5** | 0.99379 | 0.99396 | 0.99413 | 0.99430 | 0.99446 | 0.99461 | 0.99477 | 0.99492 | 0.99506 | 0.99520 |
| **2.6** | 0.99534 | 0.99547 | 0.99560 | 0.99573 | 0.99585 | 0.99598 | 0.99609 | 0.99621 | 0.99632 | 0.99643 |
| **2.7** | 0.99653 | 0.99664 | 0.99674 | 0.99683 | 0.99693 | 0.99702 | 0.99711 | 0.99720 | 0.99728 | 0.99736 |
| **2.8** | 0.99744 | 0.99752 | 0.99760 | 0.99767 | 0.99774 | 0.99781 | 0.99788 | 0.99795 | 0.99801 | 0.99807 |
| **2.9** | 0.99813 | 0.99819 | 0.99825 | 0.99831 | 0.99836 | 0.99841 | 0.99846 | 0.99851 | 0.99856 | 0.99861 |
| **3.0** | 0.99865 | 0.99869 | 0.99874 | 0.99878 | 0.99882 | 0.99886 | 0.99889 | 0.99893 | 0.99896 | 0.99900 |
| **3.1** | 0.99903 | 0.99906 | 0.99910 | 0.99913 | 0.99916 | 0.99918 | 0.99921 | 0.99924 | 0.99926 | 0.99929 |
| **3.2** | 0.99931 | 0.99934 | 0.99936 | 0.99938 | 0.99940 | 0.99942 | 0.99944 | 0.99946 | 0.99948 | 0.99950 |
| **3.3** | 0.99952 | 0.99953 | 0.99955 | 0.99957 | 0.99958 | 0.99960 | 0.99961 | 0.99962 | 0.99964 | 0.99965 |
| **3.4** | 0.99966 | 0.99968 | 0.99969 | 0.99970 | 0.99971 | 0.99972 | 0.99973 | 0.99974 | 0.99975 | 0.99976 |
| **3.5** | 0.99977 | 0.99978 | 0.99978 | 0.99979 | 0.99980 | 0.99981 | 0.99981 | 0.99982 | 0.99983 | 0.99983 |
| **3.6** | 0.99984 | 0.99985 | 0.99985 | 0.99986 | 0.99986 | 0.99987 | 0.99987 | 0.99988 | 0.99988 | 0.99989 |
| **3.7** | 0.99989 | 0.99990 | 0.99990 | 0.99990 | 0.99991 | 0.99991 | 0.99992 | 0.99992 | 0.99992 | 0.99992 |
| **3.8** | 0.99993 | 0.99993 | 0.99993 | 0.99994 | 0.99994 | 0.99994 | 0.99994 | 0.99995 | 0.99995 | 0.99995 |
| **3.9** | 0.99995 | 0.99995 | 0.99996 | 0.99996 | 0.99996 | 0.99996 | 0.99996 | 0.99996 | 0.99997 | 0.99997 |
| **4.0** | 0.99997 | 0.99997 | 0.99997 | 0.99997 | 0.99997 | 0.99997 | 0.99998 | 0.99998 | 0.99998 | 0.99998 |
| **4.1** | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99998 | 0.99999 | 0.99999 |