Statistics Ph.D. Qualifying Exam: Part II

October 13, 2023

Student Name: ______________________________

Student UID: ______________________________

1. Answer 8 out of 12 problems. Mark the problems you selected in the following table.

Problem	1	2	3	4	5	6	7	8	9	10	11	12
Selected												
Scores												

2. Write your answer right after each problem selected, attach more pages if necessary. Assemble your work in right order.

3. You can use the N(0,1) distribution table as attached.

1. Let X and Y be independent Poisson random variables with parameters θ and λ, respectively. Show that the conditional distribution of $Y|(X+Y)$ is binomial with the probability of success $\lambda/(\theta+\lambda)$.

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution whose pdf is

$$f(x;\theta) = \frac{\log(\theta)}{\theta - 1}\theta^x, \quad 0 < x < 1,\, \theta > 1.$$

Is there a function of θ, say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramér-Rao lower bound? If so, find it. If not, show why not.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ population, and suppose that the prior distribution on θ is $N(\mu, \tau^2)$. Here we assume that σ^2, μ, and τ^2 are all known.

   (a) Find the joint pdf of $\bar{X}$ and θ, where $\bar{X}$ is the mean of the random sample.

   (b) Find the posterior distribution of θ given $\bar{X}$.

   (c) If the squared error loss function is used, find the Bayes estimator of θ.

4. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be two independent random samples from beta$(\mu, 1)$ and beta$(\theta, 1)$ populations, respectively.

   (a) Find a likelihood ratio test of

$$H_0 : \mu = \theta \quad \text{versus} \quad H_1 : \mu \neq \theta.$$

   (b) Show that the test in part (a) can be based on the statistic

$$T = \frac{\sum_{i=1}^{n} \log X_i}{\sum_{i=1}^{n} \log X_i + \sum_{j=1}^{m} \log Y_j}.$$

5. Suppose X_1, X_2, ... are jointly continuous and independently distributed with marginal pdf $f(x)$, where each X_i represents annual rainfall at a given location.

   (a) Find the distribution for the number of years until the first year's rainfall (X_1) is exceeded for the first time.

   (b) Show that the mean number of years until X_1 is exceeded for the first time is infinite.

6. Let X_1, X_2, ..., X_n be a random sample from a population with pdf

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0$$

(a) Is $\sum X_i$ sufficient for θ? Explain.

(b) Find a complete sufficient statistic for θ.

7. Let X_1, X_2, ..., X_n be a random sample from the uniform(θ, $\theta + 1$) distribution. To test $H_0 : \theta = 0$ v.s. $H_1 : \theta > 0$, use the test

$$\text{reject } H_0 \text{ if } Y_n \geq 1 \text{ or } Y_1 \geq k,$$

where k is a constant, $Y_1 = min\{X_1, X_2, \ldots, X_n\}$, $Y_n = max\{X_1, X_2, \ldots, X_n\}$.

(a) Determine k so that the test will have size α.

(b) Find an expression for the power function of the test in part (a).

(c) Prove that the test is UMP size α.

8. Suppose (X, Y) have a trinomial distribution with parameters n, θ_1, θ_2, where n is the numbers of trials, and $0 \leq \theta_1 + \theta_2 \leq 1$. That is,

$$P(X = x, Y = y|\theta_1, \theta_2) = \frac{n!}{x!y!(n-x-y)!}\theta_1^x \theta_2^y (1 - \theta_1 - \theta_2)^{n-x-y}.$$

Suppose we put the Dirichlet density

$$\pi(\theta_1, \theta_2) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\theta_1^{a-1}\theta_2^{b-1}(1-\theta_1-\theta_2)^{c-1} \quad 0 \leq \theta_1 \leq 1, 0 \leq \theta_2 \leq 1; 0 \leq \theta_1+\theta_2 \leq 1,$$

as prior for (θ_1, θ_2).

(a) Let r, s, t be positive. Find $E\theta_1^r \theta_2^s (1 - \theta_1 - \theta_2)^t$.

(b) Find the posterior distribution of $((\theta_1, \theta_2)|X = x, Y = y)$. Is the Dirichlet distribution a conjugate prior for this problem?

(c) For any function h of θ_1 and θ_2, defining the Bayes estimator of $h(\theta_1, \theta_2)$ by $d_B(\mathbf{X}) = E(h(\theta_1, \theta_2)|\mathbf{X})$, find the Bayes estimators of θ_2, and $\theta_1\theta_2(1 - \theta_1 - \theta_2)$.

9. Let X and Y be random variables such that $Y|X = x \sim \text{Poisson}(\lambda x)$, and X has density

$$f_X(x) = \frac{\theta^\theta x^{\theta-1} e^{-\theta x}}{\Gamma(\theta)}, \quad x \geq 0.$$

Prove that

(a) $E(Y) = \lambda$ and $Var(Y) = \lambda + \theta\lambda^2$.

(b) Y has density

$$f_Y(y; \lambda) = \frac{\Gamma(\theta + y)\lambda^y \theta^\theta}{\Gamma(\theta) y! (\theta + \lambda)^{\theta+y}}, \quad y = 0, 1, 2, \ldots$$

10. Let $X_1, \ldots, X_m$ be a random sample from $N(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_n$ a random sample from $N(\mu_2, 4\sigma^2)$.

   (a) Obtain maximum likelihood estimators of μ_1, μ_2, and σ^2.

   (b) Derive the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2$ vs. $H_1 : \mu_1 \neq \mu_2$. What is the sampling distribution of your test statistic under H_0?

11. Suppose that $X_1, \ldots, X_n$ form a random sample from a Poisson distribution with unknown mean θ. Let $g(\theta) = P(X_1 + X_2 = 1)$.

   (a) Find $g(\theta)$.

   (b) Find an unbiased estimator of $g(\theta)$.

   (c) Find maximum likelihood estimator of $g(\theta)$.

   (d) Find a uniform minimum variance unbiased estimator (UMVUE) of $g(\theta)$.

12. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independently and identically distributed as (X, Y), where (X, Y) follows a bivariate normal distribution with means $E(X) = \mu_1$ and $E(Y) = \mu_2$ and with the variances $Var(X) = \sigma_1^2$ and $Var(Y) = \sigma_2^2$ respectively. Assume that X is un-correlated with Y.

    (a) Derive the likelihood ratio test (LRT) for testing $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative hypothesis $H_1 : \sigma_1^2 \neq \sigma_2^2$.

    (b) Derive the probability distribution of your test statistic under H_1. What is the power function of the LRT test?

Table of P(Z < z), Z ~N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998
4.1	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99998	0.99999	0.99999