Department of Mathematical Sciences
PhD Qualifying Exam

Name: ______________________________

Examination:	**Real Variables I & II**		
Date of Exam:	01/09/2023	**Duration:**	180 Minutes
No. pages:	12	**No. problems:**	8

Instructions:

This exam consists of 8 problems, grouped by semester: Problems 1-4 cover material from the first semester, while Problems 5-8 are taken from the second semester. However, you may use techniques from either semester to work on the problems. When you are asked to prove a known result, make sure you provide the key arguments needed for the proof.

Please observe the following rules when choosing the problems you intend to work on:

- **Solve at least two problems from among Problems 1-4.**
- **Solve at least two problems from among Problems 5-8.**
- **Solve a total of at least 5 problems.** Hence, you may choose your 5th problem freely.
- **Provide clear and concise justifications for your answers.**

Note: Completeness of solutions is an important factor in earning a passing grade.
Write your solutions in the space provided. If you need more space, three additional, empty pages are attached at the end of the exam. You have 3 hours to finish your work. Good luck!

Please do not write below this line.

Problem	1	2	3	4	5	6	7	8	Total
Score									
Complete									

	Pass	Fail
MS		
PhD		

Problem 1:

Denote the Lebesgue measure on $\mathbb{R}$ by m.

(a) State the Dominated Convergence Theorem of Lebesgue Integration.

(b) Let $(f)_n$ be a sequence of real-valued, measurable functions on $[0,1]$ which converges pointwise a.e. to the real-valued function f. Prove: $\lim_n \int_{[0,1]} \frac{f_n}{1+f_n^2}\, dm = \int_{[0,1]} \frac{f}{1+f^2}\, dm < \infty$.

Problem 2:

(a) Suppose that $1 \leq r \leq p \leq s \leq \infty$. Prove: $L^r(\mathbb{R}) \cap L^s(\mathbb{R}) \subset L^p(\mathbb{R})$.

(b) Prove or disprove: For every measurable set $S \subset \mathbb{R}$ and $1 \leq p \leq q \leq \infty$, $L^q(S) \subset L^p(S)$.

Problem 3:

(a) Let $f : [0, 1] \to \mathbb{R}$ be an absolutely continuous function.
Prove: f is Lipschitz continuous if and only if $f' \in L^\infty([0, 1])$.

(b) Let $f : [0, 1] \to \mathbb{R}$ be an absolutely continuous, increasing function.
Prove: If $E \subset [0, 1]$ has (Lebesgue) measure 0, then so does $f(E)$.

Problem 4:

Denote the Lebesgue measure on $\mathbb{R}$ by m.

Let (f_n) be a sequence of integrable functions on $\mathbb{R}$. Suppose there exist an integrable function g and a measurable function h such that

- (f_n) converges to g in mean, i.e. $\lim_n \int_{\mathbb{R}} |f_n - g| \, dm = 0$,
- (f_n) converges to h in measure.

Prove: $g = h$ a.e.

Note: One way to approach this problem is to use that convergence in mean and convergence in measure are related to pointwise a.e. convergence. How are they related? Prove this relationship or clearly cite the relevant theorems.

Problem 5:

(a) State the Uniform Boundedness Principle.

(b) Let $\mathcal{X}$ be a Banach space and (T_n) a sequence of bounded linear operators on $\mathcal{X}$ such that $\lim_n \sum_{k=1}^{n} T_k(x)$ exists for every $x \in X$. Prove: The sequence (T_n) is bounded in the operator norm.

Problem 6:

Denote the Lebesgue measure on $\mathbb{R}$ by m. Let f and g be functions in $L^1(\mathbb{R})$.

(a) Prove: For a.e. $x \in \mathbb{R}$, the function h_x, defined for $y \in \mathbb{R}$ by $h_x(y) = f(x-y)\,g(y)$, belongs to $L^1(\mathbb{R})$.

(b) Prove: The function $(f * g)$, defined for a.e. $x \in \mathbb{R}$ by $(f * g)(x) = \int_{\mathbb{R}} f(x-y)\,g(y)\,dm(y)$, belongs to $L^1(\mathbb{R})$.

(c) Prove: $\|(f * g)\|_1 \le \|f\|_1 \, \|g\|_1$

Problem 7:

(a) State the Closed Graph Theorem.

(b) Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Suppose $A, B : \mathcal{H} \to \mathcal{H}$ are linear operators such that for all $x, y \in \mathcal{H}$, $\langle x, Ay \rangle = \langle Bx, y \rangle$. Prove: A and B are bounded.

Problem 8:

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{Y} \subset \mathcal{H}$ a closed subspace.

(a) Prove: If, for given $x \in \mathcal{H}$, $y \in \mathcal{Y}$ is such that, among all vectors in $\mathcal{Y}$, y has minimal distance to x, then $x - y \in \mathcal{Y}^{\perp}$.
(Note: $\mathcal{Y}^{\perp}$ is the subspace of all $u \in \mathcal{H}$ such that $\langle u, v \rangle = 0$ for all $v \in \mathcal{Y}$.)

(b) Prove: If $P : \mathcal{H} \to \mathcal{H}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{Y}$, then $P(I - P) = 0$.
(Note: $I : \mathcal{H} \to \mathcal{H}$ denotes the identity operator with $I(x) = x$ for all $x \in \mathcal{H}$.)

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