ALTERNATING KNOT DIAGRAMS, EULER CIRCUITS AND THE INTERLACE POLYNOMIAL

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ABSTRACT. We show that two classical theorems in graph theory and a simple result concerning the interlace polynomial imply that if K is a reduced alternating link diagram with $n \geq 2$ crossings then the determinant of K is at least n. This gives a particularly simple proof of the fact that reduced alternating links are nontrivial.

Tait's conjectures concerning alternating knot diagrams remained open for over 100 years, and were proved only a few years ago by Kauffman [7], Murasugi [9] and Thistlethwaite [11] with the aid of the Jones polynomial. The weak form of one of these conjectures, namely that every knot having a reduced alternating diagram with at least one crossing is nontrivial, was first proved by Bankwitz [5] in 1930; more recently, Menasco and Thistlethwaite [8] and Andersson [2] published simpler proofs. Our aim in this note is to point out that this result on alternating knots is closely related to two fundamental theorems in graph theory and a simple extremal property of the recently introduced interlace polynomial. This relationship gives a very simple combinatorial proof of the assertion that if K is a reduced alternating link diagram with $n \geq 2$ crossings then the determinant $\det K$ of K is at least n. Since the determinant is an ambient isotopy invariant of link diagrams, this gives a particularly simple proof of the fact that alternating links are nontrivial.

Let us start by recalling some basic definitions and results concerning directed multigraphs, or *digraphs* as we shall call them. Let G be a digraph with vertex set $\{v_1, \dots, v_n\}$, with a_{ij} edges from v_i to v_j. The *outdegree* of a vertex v_i is $d^+(v_i) = \sum_{j=1}^n a_{ij}$, and the *indegree* of v_i is $d^-(v_i) = \sum_{j=1}^n a_{ji}$. The *adjacency matrix* of G is $A = A(G) = (a_{ij})$, and its *(combinatorial) Laplacian* is the matrix $L = L(G) = (\ell_{ij}) = D - A$, where $D = (d_{ij})$ is the diagonal matrix with $d_{ii} = d^+(v_i)$. We shall write $\ell_i(G)$ for the first cofactor of $L(G)$ belonging to ℓ_{ii}.

A spanning tree T of G is *oriented towards* v_i if for every edge $\overrightarrow{v_j v_k} \in E(T)$, the vertex v_k is on the (unique) path in T from v_j to v_i. We shall write $t_i(G)$ for the number of spanning trees of G oriented towards v_i. In this notation, the classical matrix-tree theorem for digraphs (see, e.g., [6, p. 58, Theorem 14]) states that

$$t_i(G) = \ell_i(G). \tag{1}$$

The digraph G is *Eulerian* if it has an (oriented) Euler circuit, i.e., if it is connected and $d^+(v_i) = d^-(v_i)$ for every i. Let $s(G)$ be the number of Euler circuits of G. Then the BEST theorem of de Bruijn, van Aardenne-Ehrenfest, Smith and Tutte (see [1], and also [6, p. 19, Theorem 13]) states that

$$s(G) = t_i(G) \prod_{j=1}^{n} (d^+(v_j) - 1)!. \tag{2}$$

In particular, if G is a 2-in 2-out digraph, i.e., $d^+(v_i) = d^-(v_i) = 2$ for every i, then equations (1) and (2) imply that

$$s(G) = t_i(G) = \ell_i(G) \tag{3}$$

for every i.

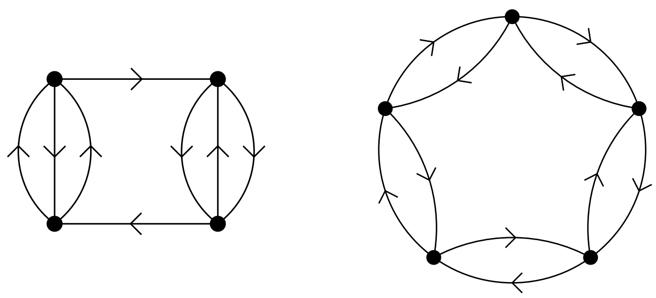
For an Euler circuit C of G, two vertices v_i and v_j are *interlaced* in C if they appear on C in the order $\dots v_i \dots v_j \dots v_i \dots v_j \dots$. Read and Rosenstiehl [10] defined the *interlace graph* $H = H(C)$ of C as the graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set

$$\{v_iv_j : v_i \text{ and } v_j \text{ are interlaced in } C\}.$$

The graph H is also said to be an *interlace graph of the digraph* G. Recently, Arratia, Bollobás and Sorkin [3] defined a one-variable polynomial $q_H(x)$ of undirected graphs H, the *interlace polynomial*, such that if H is an interlace graph of G then $q_H(1) = s(G)$. One of the many properties of the interlace polynomial $q_H(x)$ proved in [4] is that if H has $n \geq 2$ vertices, none of which is isolated, then $q_H(1) \geq n$, with equality iff either $n = 4$ and H consists of two independent edges, or $n \geq 2$ and H is a star. First we shall give an immediate consequence of this simple result.

Recall that a vertex v of a graph G is an *articulation vertex* if G is the union of two nontrivial graphs with only the vertex v in common. In particular, a vertex incident with a loop is an articulation vertex.

Theorem 1. *Let G be a connected 2-in 2-out digraph with $n \geq 2$ vertices, whose underlying multigraph has no articulation vertices. Then $s(G) \geq n$, with equality if and only if either $n = 4$ and G is the digraph shown in Fig. 1(a), or $n \geq 2$ and G is the alternately oriented double cycle with n vertices, as in Fig. 1(b).*



(a) (b)

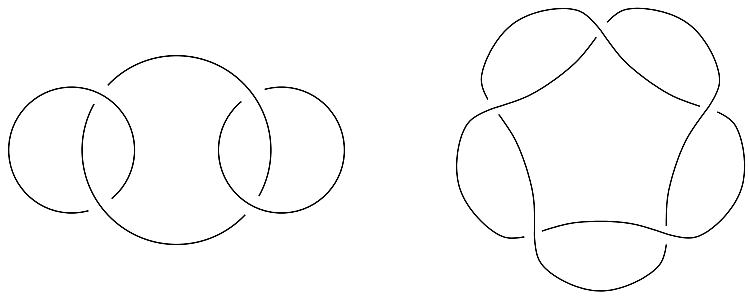
FIGURE 1. The extremal digraph for $n = 4$ and the alternately oriented double 5-cycle.

Proof. Let H be the interlace graph of an Euler circuit C of G. If v is an isolated vertex of H then v is interlaced with no other vertex of G in C. Thus v splits C into two circuits C_1 and C_2 so that every vertex $w \neq v$ of G is visited either twice by C_1, or twice by C_2. This implies that v is an articulation vertex of G. Thus H has no isolated vertices, so $s(G) = q_H(1) \geq n$, with equality iff either $H = 2K_2$ or H is a star. In the first case, G is the digraph shown in Fig. 1(a). Also, by the definition of the interlace graph, H is a star iff in C one vertex is interlaced with every vertex, but no other two vertices are interlaced. In the second case G is thus an alternately oriented double cycle. □

Let us recall the definition of the Alexander polynomial of a link diagram. First, a *strand* of a link diagram is an arc of the diagram from an undercrossing to an undercrossing, with only overcrossings in its interior. Thus a link diagram with n crossings has precisely n strands. Let K be a connected oriented link diagram with crossings $v_1, \ldots, v_n$ and strands $s_1, \ldots, s_n$, $n \geq 1$. The *Alexander matrix* $M_K(t) = (m_{ij})$ of K is the n by n matrix defined as follows. Suppose that, at a crossing v_ℓ, strand s_i passes over strands s_j and s_k in such a way that if s_i is rotated counterclockwise to cover s_j and s_k, then s_i is oriented from s_j to s_k. If s_i, s_j and s_k are distinct then $m_{\ell,i} = 1 - t$, $m_{\ell,j} = -1$, $m_{\ell,k} = t$, and all other entries in row ℓ are 0; if two or more of the strands are the same then we add the corresponding entries. The *Alexander polynomial* $A_K(t)$ of K is the determinant of the matrix obtained from $M_K(t)$ by deleting the first row and first column. (For $n = 0$ and 1 we take $A_K(t) = 1$.) Also, the *determinant* of K is $\det K = |A_K(-1)|$. In general, the Alexander polynomial of a link depends on the diagram and on the particular numbering chosen. However, up to a factor $\pm t^k$, it is an ambient isotopy invariant, i.e., it is independent of the particular diagram and of the numbering used. In particular, the determinant is an invariant of ambient isotopy. (In fact, the invariance of the determinant is even easier to see than that of the Alexander polynomial.)

A link diagram K defines a 4-regular plane multigraph, the *universe* of K. A crossing of K is *nugatory* if the corresponding vertex of the universe is an articulation vertex, and a diagram is *reduced* if it is connected and has no nugatory crossings.

Theorem 2. *Let K be a reduced alternating link diagram with $n \geq 1$ crossings. Then $\det K \geq n$, with equality if and only if either $n = 4$ and K is the link diagram with three components shown in Fig. 2(a), or $n \geq 2$ and K is the standard diagram of a $(2, n)$-torus link, as in Fig. 2(b). In particular, if K is a reduced alternating diagram with at least one crossing then K is nontrivial.*



(*a*) (*b*)

Figure 2. Extremal link diagrams.

Proof. Since K is alternating, each strand goes over precisely one crossing. In particular, we may assume that the crossings are $v_1, \ldots, v_n$, the strands $s_1, \ldots, s_n$, and that strand s_i goes over crossing v_i. For each strand s_i passing over strands s_j and s_k, send directed edges from v_i to v_j and v_k. In this way we obtain a 2-in 2-out

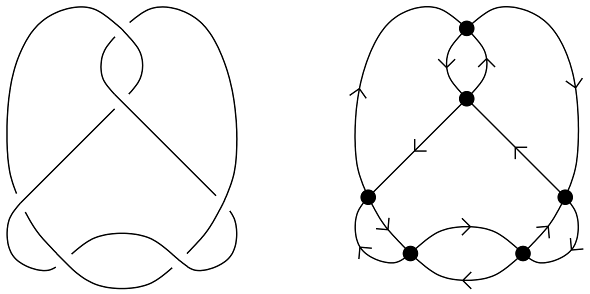


FIGURE 3. The 2-in 2-out digraph of an alternating link diagram.

digraph $G = G(K)$ on the universe of K, as in Fig. 3. As K is reduced, $n \geq 2$, and the multigraph underlying G has no articulation vertices. The Laplacian $L(G)$ of G is precisely the Alexander matrix $M_K(t)$ of K with $t = -1$. In particular, the Alexander polynomial $A_K(t)$ obtained from this representation of K, namely, the determinant of the matrix obtained from $M_K(t)$ by deleting its first row and first column, satisfies $A_K(-1) = \ell_1(G)$. Consequently, by (3) we have

$$|A_K(-1)| = s(G),$$

so the result follows from Theorem 1. □

Let K be a reduced alternating knot diagram with at least one crossing. As remarked in [2], if p is an odd prime dividing $\det K$, then K can be coloured mod p. In particular, as the determinant is always odd, if it is at least 2 then an elementary colouring argument shows that the knot is nontrivial, without any reference to the ambient isotopy invariance of the Alexander polynomial. The results in this paper arose from our failed attempts at understanding the proof in [2] that $\det K \geq 2$ for a reduced alternating diagram with at least one crossing.

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