Independence Densities of Graphs and Hypergraphs

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Joint work with Béla Bollobás, and Karen Gunderson

AMS Sectional Meeting, North Carolina State University, Raleigh, NC

The independence density

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The *independence density* of a *countably infinite* hypergraph $\mathcal{H}$ is the limit of independence densities of a chain of finite induced subhypergraphs

$$\mathrm{id}(\mathcal{H}) = \lim_{n\to\infty} \mathrm{id}(\mathcal{H}_n),$$

where $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{H}_3 \subseteq \dots$, $\mathcal{H}_n = \mathcal{H}[V_n]$, $\mathcal{H} = \bigcup \mathcal{H}_n$.

The independence density

The independence density has been studied by Bonato, Brown, Kemkes, and Prałat, who, in particular, noted that the above limit for countably infinite graphs exists and is independent of the choice of chain $\{\mathcal{H}_n\}_n$. Indeed, it is an easy consequence of the following.

Observation

If $\mathcal{H}$ is an induced subhypergraph of $\mathcal{H}'$ then $\mathrm{id}(\mathcal{H}) \geq \mathrm{id}(\mathcal{H}')$.

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Proof.

Taking a subset S of $V(\mathcal{H}')$, the condition that S is independent in $\mathcal{H}'$ is stronger than the condition that $S \cap V(\mathcal{H})$ is independent in $\mathcal{H}$. □

Probabilistic Interpretation

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All our results on id generalize to id_p.

Possible independence densities

For graphs:

Conjecture (Bonato, Brown, Kemkes, Prałat 2011)

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For graphs and hypergraphs:

Theorem (Bonato, Brown, Mitsche, Prałat 2014)

If the edges of $\mathcal{H}$ are of bounded size then $\mathrm{id}(\mathcal{H})$ is rational. If the edges of $\mathcal{H}$ are of unbounded size then $\mathrm{id}(\mathcal{H})$ can be any value in $[0, 1]$.

Possible independence densities

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$\{\mathrm{id}(\mathcal{H}) : \mathcal{H} \textit{ countable, } r(\mathcal{H}) \le k\} = \{0\} \cup \{\mathrm{id}(\mathcal{H}) : \mathcal{H} \textit{ finite, } r(\mathcal{H}) \le k\}$

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Moreover, both results also hold for $\mathrm{id}_p(\mathcal{H})$.

A counterexample to a stronger claim

It would be nice to replace "$r(\mathcal{H}) \leq k$" with "k-uniform", i.e., require all edges to be of size exactly k. Unfortunately this does not hold for $\mathrm{id}_p(\mathcal{H})$ even when $k = 2$.

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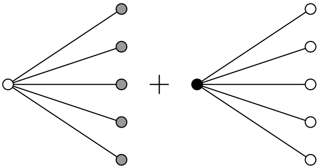
Example: let $K_{1,\infty}$ be the "infinite star".

$$\mathrm{id}_p(K_{1,\infty}) = \lim_{n\to\infty} \mathrm{id}_p(K_{1,n}) = \lim_{n\to\infty} (1 - p + p(1-p)^n) = 1 - p.$$



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It remains open whether or not we need edges of size $< k$ for $p = 1/2$ in the hypergraph case for $k > 2$.

Shrinking edges

Clearly if $E, E' \in \mathcal{H}$ and $E \subseteq E'$, then $\mathrm{id}(\mathcal{H}) = \mathrm{id}(\mathcal{H} \setminus \{E'\})$. So we may assume $\mathcal{H}$ is an *antichain*.

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Suppose X is a set of vertices and $X \cup Y_i \in \mathcal{H}$ for $i = 1, \dots, m$, with $X, Y_1, \dots, Y_m$ pairwise disjoint, then

$$\mathrm{id}(\mathcal{H} \cup \{X\}) \leq \mathrm{id}(\mathcal{H}) \leq \mathrm{id}(\mathcal{H} \cup \{X\}) + 2^{-|X|}(1 - 2^{|X|-k})^m.$$

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In particular, if X can be extended to edges of $\mathcal{H}$ in infinitely many disjoint ways, then

$$\mathrm{id}(\mathcal{H} \cup \{X\}) = \mathrm{id}(\mathcal{H} \setminus \{E : E \supset X\} \cup \{X\}) = \mathrm{id}(\mathcal{H}).$$

Main Lemma

Define the *matching number* $\mu(\mathcal{H})$ as the maximum number of pairwise disjoint edges of $\mathcal{H}$.

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Lemma

Let x_n be independence densities of countable hypergraphs with rank at most k, and $x_n \to x > 0$. Then there exists a finite hypergraph $\mathcal{H}$ on a vertex set V, an increasing sequence n_i and countable hypergraphs $\mathcal{H}_i$ such that

- $\mathrm{id}(\mathcal{H}_i) = x_{n_i}$,
- $\mathcal{H}_i[V] = \mathcal{H}$,
- *if $E \in \mathcal{H}_i$ with $E \not\subseteq V$, then there exists $A \subseteq E \cap V$ with $\mu(F \subseteq V^c : F \cup A \in \mathcal{H}_i) \geq i$.*

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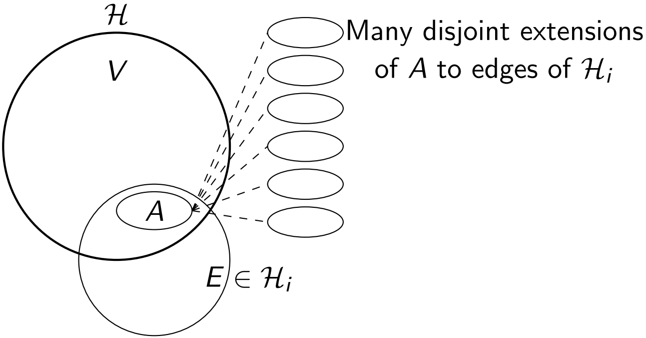

Proof of main lemma

Idea is to inductively construct for each $j = 1, \ldots, k$ an $\mathcal{H}$, V, sequence n_i, and countable $\mathcal{H}_i$, such that

- $\mathrm{id}(\mathcal{H}_i) = x_{n_i}$;
- $\mathcal{H}_i[V] = \mathcal{H}$,
- if $E \in \mathcal{H}_i$ with $E \not\subseteq V$ **and** $|E \cap V| < j$ then there exists $A \subseteq E \cap V$ with $\mu(F \subseteq V^c : F \cup A \in \mathcal{H}_i) \geq i$.

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Result then follows from the $j = k$ case when $r(\mathcal{H}) \leq k$.

Basic idea is if this fails, go to a subsequence where the $\mu(\ldots)$'s achieve their limsup, then to a subsequence where these max matchings $+$ V induces a fixed hypergraph $\mathcal{H}'$. This is our new $\mathcal{H}$.

Proof of main results

We aim to show that if $\mathcal{H}$ is countable then either $\mathrm{id}(\mathcal{H}) = 0$ or $\mathrm{id}(\mathcal{H}) = \mathrm{id}(\mathcal{H}_0)$ for some finite $\mathcal{H}_0$ and furthermore the set of all possible independence densities is closed and has no infinite increasing sequence.

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It is enough to show that if $x_n = \mathrm{id}(\mathcal{H}_n)$ and $x_n \to x > 0$ then x is an independence density of a finite hypergraph and the sequence x_n cannot be strictly increasing.

Proof of main results

Using Lemma we may assume there is a finite $\mathcal{H}$ on vertex set V and countable $\mathcal{H}_i$ with

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Set $\mathcal{H}' = \mathcal{H} \cup \mathcal{A}$.
It can be checked that $\mathrm{id}(\mathcal{H}') = x \leq x_{n_i}$, proving both results.

Open Questions

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Thank you!