Longest paths in circular arc graphs
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Abstract. We show that all maximum length paths in a connected circular arc graph have non–empty intersection.

Keywords: interval graph, circular arc graph, maximum length path

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1. Introduction

Every two maximum length paths in a connected graph have a common vertex. Gallai in [1] asked whether all maximum length paths share a common vertex of the graph. This perfect "Helly–property" on longest paths is not true in general. The first counter–example was constructed by H. Walther, and the smallest known counter–example in Fig. 1 is due to Zamfirescu [6].

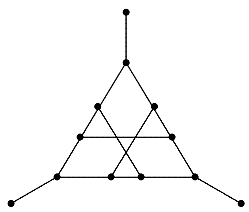

Figure 1. No vertex covers all longest paths

These and many further examples in Skupień [4] all contain induced cycles longer than three with no chord. In other words, the known counter–examples are not chordal graphs.

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We have not been able to determine whether the longest paths of every chordal graph have the Helly–property envisioned by Gallai. However, Klavžar and Petkovšek in [3] observed that this is true in a connected split graph (split graphs are chordal graphs whose complement is also chordal). In addition we shall prove that Gallai's question has an affirmative answer for interval graphs, another distinguished subfamily of chordal graphs.

A graph $G = (V, E)$ is an *interval graph*, if there exists a mapping ι of its vertex set V into a collection of intervals of the real line such that, for every $u, w \in V$, uw is an edge of G if and only if $\iota(u) \cap \iota(w) \neq \emptyset$.

The interval representation approach was successfully applied to a non–chordal extension of interval graphs leading to the main theorem of our paper.

A graph $G = (V, E)$ is a *circular arc graph*, if there exists a mapping α of its vertex set V into a collection of arcs of a circle such that, for every $u, w \in V$, uw is an edge of G if and only if $\alpha(u) \cap \alpha(w) \neq \emptyset$.

THEOREM 1.1. *All maximum length paths of a connected circular arc graph have non–empty intersection.*

2. Intervals

Given a finite collection $\mathcal{F}$ of (open) intervals on the real line, $\mathcal{C} = (I_1, \ldots, I_t)$ is called a t–chain in $\mathcal{F}$ if $I_k \in \mathcal{F}$ are distinct for $1 \leq k \leq t$, and $I_k \cap I_{k+1} \neq \emptyset$, for every $1 \leq k \leq t-1$. A chain containing the maximum number of intervals is called a longest chain of $\mathcal{F}$. Note that the chains in $\mathcal{F}$ correspond to the paths in the intersection graph of $\mathcal{F}$. The *support* of a chain $\mathcal{C}$ is defined as the set

$$\operatorname{Supp}\mathcal{C} = I_1 \cup (I_2 \cap I_3) \cup \ldots \cup (I_{t-2} \cap I_{t-1}) \cup I_t \ .$$

Observe that for a longest chain $\mathcal{C}$ and an interval $J \in \mathcal{F}$, we have $J \in \mathcal{C}$ if and only if $J \cap \operatorname{Supp}\mathcal{C} \neq \emptyset$. In fact, if $J \notin \mathcal{F}$ and $J \cap \operatorname{Supp}\mathcal{C} \neq \emptyset$ then J could be inserted into the chain between two consecutive intervals or at the end of $\mathcal{C}$, thus contradicting its maximality.

THEOREM 2.1. *In every finite collection of intervals on the real line with connected union there is an interval belonging to all longest chains.*

Proof. Assume that t is the maximum number of intervals of a chain in the collection $\mathcal{F}$, and let N be the number of t–chains in $\mathcal{F}$. We show that for each $n = 2, \ldots, N$, every n of the t–chains in $\mathcal{F}$ have a common interval. The proof is induction on n. For the case $n = 2$, observe that the intersection graph of $\mathcal{F}$ is connected, by the assumption, and in a connected graph any two longest paths have a common vertex. In other words, every two t–chains in $\mathcal{F}$ share a common interval.

Next let $n \geq 3$, and let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be n longest chains in $\mathcal{F}$. Set $\mathcal{C}_k = (I_1^k, \ldots, I_t^k)$, $1 \leq k \leq n$. We assume that for every $1 \leq k \leq n$, there is an interval $J_k \in \mathcal{F}$ such that

$$J_k \in \bigcap \{\mathcal{C}_i \ : \ 1 \leq i \leq n \text{ and } i \neq k\} \ .$$

Clearly we may assume that $J_k \cap \operatorname{Supp} \mathcal{C}_k = \emptyset$, otherwise $J_k \in \mathcal{C}_k$, and the proof is complete. We show that one can assume even the stronger condition $J_k \cap \operatorname{Conv}(\operatorname{Supp} \mathcal{C}_k) = \emptyset$, where $\operatorname{Conv}(\operatorname{Supp} \mathcal{C}_k)$ is the smallest interval containing $\operatorname{Supp} \mathcal{C}_k$.

Assuming that $J_k \cap \operatorname{Conv}(\operatorname{Supp} \mathcal{C}_k) \neq \emptyset$, there exist points of $\operatorname{Supp} \mathcal{C}_k$ on both sides of the interval J_k. Therefore there exists $1 < r < t$ such that J_k is between the intervals $I_{r-1}^k \cap I_r^k$ and $I_r^k \cap I_{r+1}^k$. In particular, we have $I_r^k \supseteq J_k$. Then for every $1 \leq i \leq n$ and $i \neq k$, we have $J_k \in \mathcal{C}_i$ which implies $I_r^k \cap \operatorname{Supp} \mathcal{C}_i \neq \emptyset$. In addition, $I_r^k \in \mathcal{C}_k$, thus we conclude that $I_r^k \in \bigcap \{\mathcal{C}_i \ : \ 1 \leq i \leq n\}$.

Now we assume that $J_k \cap \operatorname{Conv}(\operatorname{Supp} \mathcal{C}_k) = \emptyset$, for every $1 \leq k \leq n$. This means that either $J_k < \operatorname{Supp} \mathcal{C}_k$ or $\operatorname{Supp} \mathcal{C}_k < J_k$, where the inequality means left–to–right ordering of disjoint intervals in the real line. Because $n \geq 3$, we have the same alternative for two indices. We shall assume that $\operatorname{Supp} \mathcal{C}_1 < J_1$ and $\operatorname{Supp} \mathcal{C}_2 < J_2$ (w.l.o.g. since one can freely reverse the left–to–right ordering on the real line). Then we obtain that either $\operatorname{Supp} \mathcal{C}_1 < J_2$ or $\operatorname{Supp} \mathcal{C}_2 < J_1$. Either of them contradicts the induction hypothesis $J_2 \in \mathcal{C}_1$ or $J_1 \in \mathcal{C}_2$. □

Note that Theorem 2.1 proves Theorem 1.1 in the particular case when the circular arc is an interval graph.

COROLLARY 2.2. *All maximum length paths of a connected interval graph have non–empty intersection.* □

When extending this theorem to circular arc graphs we will use a lemma motivated (and implicitly included) by Keil's work on the hamiltonicity of interval graphs [2].

LEMMA 2.3. *Let $X = \{x_1, x_2, \ldots, x_{t+1}\}$ be a set of real numbers, and let $J_1, J_2, \ldots, J_t$ be a sequence of open real intervals with $x_k, x_{k+1} \in J_k$, for every $1 \leq k \leq t$. If $x_{i_1} < x_{i_2} < \ldots < x_{i_{t+1}}$ are the elements of X in increasing order, then the intervals have a permutation $J_{j_1}, J_{j_2}, \ldots, J_{j_t}$ such that $x_{i_k}, x_{i_{k+1}} \in J_{j_k}$, for every $1 \leq k \leq t$.*

Proof. For every $i = 1, \dots, t$, let $J_i^* \subset J_i$ be the closed interval with endpoints x_i and x_{i+1}, so that $J_i^* = [x_i, x_{i+1}]$ if $x_i < x_{i+1}$ and $J_i^* = [x_{i+1}, x_i]$ otherwise. We shall show the existence of a bijection ψ from the set of consecutive segments $\mathcal{X} = \{[x_{i_1}, x_{i_2}], [x_{i_2}, x_{i_3}], \dots, [x_{i_t}, x_{i_{t+1}}]\}$ onto $\mathcal{J}^* = \{J_1^*, \dots, J_t^*\}$ such that $[x_{i_k}, x_{i_{k+1}}] \subseteq \psi([x_{i_k}, x_{i_{k+1}}])$, for every $1 \le k \le t$. Then the required permutation of the J_ks follows from $x_{i_k}, x_{i_{k+1}} \in \psi([x_{i_k}, x_{i_{k+1}}]) = J_{j_k}^* \subset J_{j_k}$.

To see the existence of such a bijection we shall verify Hall's condition for the bipartite graph on $\mathcal{X} \cup \mathcal{J}^*$ with an edge between vertices $[x_{i_k}, x_{i_{k+1}}] \in \mathcal{X}$ and $J_i^* \in \mathcal{J}^*$ if and only if $[x_{i_k}, x_{i_{k+1}}] \subseteq J_i^*$.

Let $S \subseteq \{1, \dots, t\}$ and let

$$X(S) = \{x_{i_k} \in X \ : \ [x_{i_k}, x_{i_{k+1}}] \subseteq J_i^* \text{ for some } i \in S\} \ .$$

Write $\bigcup_{i \in S} J_i^* = \bigcup_{j=1}^{r} [a_j, b_j]$, where the $[a_j, b_j]$ are disjoint intervals. Let $[a_j, b_j] = \bigcup_{i \in S_j} J_i^*$, for $j = 1, \dots, r$. Clearly $|X \cap [a_j, b_j]| \ge |S_j| + 1$ implying $|X(S_j)| \ge |S_j|$. Since $X(S_j)$ are disjoint, the Hall's condition is obtained from

$$\begin{aligned} &|\{[x_{i_k}, x_{i_{k+1}}] \in \mathcal{X} \ : [x_{i_k}, x_{i_{k+1}}] \subseteq \bigcup_{j \in S} J_i^*\}| \\ &= |X(S)| = \sum_{j=1}^{r} |X(S_j)| \ge \sum_{j=1}^{r} |S_j| = |S| = |\{J_i^* \in \mathcal{J}^* \ : \ i \in S\}| \ . \end{aligned}$$

Hence the required permutation does exist. □

3. Arcs

Let C be a circle and let $\mathcal{F}$ be a finite collection of open arcs of C. Note that if C has a point not in $\bigcup \mathcal{F}$, then the intersection graph of $\mathcal{F}$ is an interval graph. Thus we assume that $\bigcup \mathcal{F} = C$. Let $\mathcal{K} = \{K_0, \dots, K_{n-1}\}$ be a set of n arcs of $\mathcal{F}$ such that

(0) $C = K_0 \cup \ldots \cup K_{n-1}$,
(1) n is minimal, and
(2) each K_i is maximal, so $A \supseteq K_i$ and $A \in \mathcal{F}$ imply $A = K_i$.

It is clear that such a set $\mathcal{K}$ exists. Note that $n \ge 2$ iff no arc contains C which is assumed in the sequal. We also assume that for any two arcs $A, B \in \mathcal{F}$, the endpoints of A and B are distinct.

We order $\mathcal{K}$ so that K_{i+1} is the arc immediately clockwise from K_i. In particular, note that requirement (1) implies that K_i intersects only K_{i-1} and K_{i+1}, where the indices are taken mod n.

A path on t vertices in the intersection graph of $\mathcal{F}$ corresponds to a t–chain $\mathcal{P} = (A_1, \ldots, A_t)$ in $\mathcal{F}$. Define the *support* of $\mathcal{P}$ as the set

$$\operatorname{Supp}\mathcal{P} = A_1 \cup (A_2 \cap A_3) \cup \ldots \cup (A_{t-2} \cap A_{t-1}) \cup A_t \ .$$

Assume from now on that t is the maximum number of arcs of a chain in $\mathcal{F}$. Observe that if $\mathcal{P}$ is a longest chain in $\mathcal{F}$, then $A \in \mathcal{P}$ if and only if $A \cap \operatorname{Supp}\mathcal{P} \neq \emptyset$.

LEMMA 3.1. *If $\mathcal{P}$ is a longest chain in $\mathcal{F}$, then $\mathcal{P} \cap \mathcal{K} = \{K_i : i \in I\}$ is non–empty and I is a contiguous set of elements of $\mathbb{Z}_n$.*

Proof. By (0), the support of each longest chain $\mathcal{P}$ does intersect at least one arc of $\mathcal{K}$. Then this arc belongs to $\mathcal{P}$ and $\mathcal{P} \cap \mathcal{K} \neq \emptyset$ follows.

The lemma is obviously true for $n \leq 3$. Assume that $n \geq 4$, and suppose that $K_i, K_j, K_k, K_\ell \in \mathcal{K}$ are ordered cyclically with $K_i, K_k \in \mathcal{P}$ and $K_j, K_\ell \notin \mathcal{P}$. By the minimality of n required in (1), we have $C \setminus (K_j \cup K_\ell)$ is a union of two disjoint arcs, and $\operatorname{Supp}\mathcal{P}$ meets both arcs (since it meets both K_i and K_k). By the connectedness of $\mathcal{P}$ there exists $A \in \mathcal{P}$ meeting both components of $C \setminus (K_j \cup K_\ell)$. But then either $K_j \subset A$ or $K_\ell \subset A$, contradicting (2), the maximality of either K_j or K_ℓ. □

If $\mathcal{P}$ is a t–chain in $\mathcal{F}$, then by Lemma 3.1, we have $\mathcal{P} \cap \mathcal{K} = \{K_{a+1}, K_{a+2} \ldots, K_{b-1}\}$. Assume that $\mathcal{K} \not\subseteq \mathcal{P}$ so that $K_k \notin \mathcal{P}$ for every $k = b, b+1, \ldots, a$, (possibly $a = b$, and integers are taken mod n).

If $A \in \mathcal{P}$, then $A \not\subseteq K_k$, because otherwise, $(K_k \cap \operatorname{Supp}\mathcal{P}) \supseteq (A \cap \operatorname{Supp}\mathcal{P}) \neq \emptyset$ would imply $K_k \in \mathcal{P}$. Furthermore, $A \not\supseteq K_k$ by the maximality of K_k. Therefore, for every $A_i \in \mathcal{P}$, the set $A_i \setminus (K_a \cup K_b)$ is a non–empty arc of $C' = (K_{a+1} \cup \ldots \cup K_{b-1}) \setminus (K_a \cup K_b)$. We shall assume that the points of the arc C' are ordered clockwise. These considerations combined with Lemma 2.3 are elaborated into the following lemma.

LEMMA 3.2. *Let $\mathcal{P}$ be a longest chain in $\mathcal{F}$, and assume that $\mathcal{P} \cap \mathcal{K} = \{K_{a+1}, K_{a+2} \ldots, K_{b-1}\} \neq \mathcal{K}$. Then the arcs in $\mathcal{P}$ have a reordering into a chain $\mathcal{P}^*$ such that in this reordering*

(a) K_{a+1} preceeds K_{b-1} provided they are distinct,

(b) if A preceeds K_{b-1} then $A \not\supseteq K_{b-1} \cap K_b$,

(c) if A preceeds K_{a+1} then $A \subseteq K_a \cup K_{a+1}$.

Proof. Let $\mathcal{P} = (J_1, \ldots, J_t)$, and let $\{x_1, \ldots, x_{t+1}\} \subset \operatorname{Supp}\mathcal{P}$ be a set of distinct points such that $x_k, x_{k+1} \in J_k$, for every $1 \leq k \leq t$.

Because $K_a, K_b \notin \mathcal{P}$, each x_i belongs to the arc $C' = (K_{a+1} \cup \ldots \cup K_{b-1}) \setminus (K_a \cup K_b)$. Let $x_{i_1}, x_{i_2}, \ldots, x_{i_{t+1}}$ be the permutation of these points in clockwise order in C'. Now the permutation of the arcs of $\mathcal{P}$ given in Lemma 2.3 is another longest chain $(J_{j_1}, J_{j_2}, \ldots, J_{j_t})$ such that $[x_{i_k}, x_{i_{k+1}}] \in J_{j_k}$, for every $1 \le k \le t$ (where $[x, y]$ is the circular arc going clockwise from x to y). We show a procedure of rearranging the arcs in this chain in order to fulfill the requirements (a) – (c).

Let $p < q$, $J_{j_p} = K_{b-1}$ and $J_{j_q} = K_{a+1}$. Because the left endpoint of K_{a+1} preceeds the left endpoint of K_{b-1}, and the right endpoint of K_{a+1} preceeds the right endpoint of K_{b-1}, we have $x_{i_p} \in K_{a+1}$ and $x_{i_{q+1}} \in K_{b-1}$. This implies $[x_{i_p}, x_{i_{p+1}}]$, $[x_{i_q}, x_{i_{q+1}}] \subseteq J_{j_p} \cap J_{j_q}$. Thus it is possible to swap J_{j_p} and J_{j_q} in the chain. In this reordering (a) is satisfied.

Let $p < q$, $J_{j_p} = A$ and $J_{j_q} = K_{b-1}$. Assume that A is an arc with $A \supseteq K_{b-1} \cap K_b$. Because $A \not\supseteq K_{b-1}$, the left endpoint of K_{b-1} preceeds the left endpoint of A, and the right endpoint of K_{b-1} preceeds the right endpoint of A. Thus we have $x_{i_p} \in K_{b-1}$ and $x_{i_{q+1}} \in A$ which implies $[x_{i_p}, x_{i_{p+1}}], [x_{i_q}, x_{i_{q+1}}] \subseteq J_{j_p} \cap J_{j_q}$. Then it is possible to swap $A = J_{j_p}$ and $K_{b-1} = J_{j_q}$ in the chain. By successive swapping of such arcs A we obtain a reordering where (a) and (b) are satisfied.

Let $p < q$, $J_{j_p} = A$ and $J_{j_q} = K_{a+1}$. Assume that A is an arc with $A \not\subseteq K_a \cup K_{a+1}$. We know that $A \cap C' \neq \emptyset$, furthermore, $A \not\supseteq K_a$ and $A \not\supseteq K_{a+1}$. Therefore, the left endpoint of K_{a+1} preceeds the left endpoint of A, and the right endpoint of K_{a+1} preceeds the right endpoint of A. Thus we have $x_{i_p} \in K_{a+1}$ and $x_{i_{q+1}} \in A$ which implies $[x_{i_p}, x_{i_{p+1}}], [x_{i_q}, x_{i_{q+1}}] \subseteq J_{j_p} \cap J_{j_q}$. Then, as before, it is possible to swap $A = J_{j_p}$ and $K_{a+1} = J_{j_q}$ in the chain. By successive swapping we obtain a chain satisfying (c) as well. Thus we obtain a reordering $\mathcal{P}^*$ satisfying (a) – (c) as required. □

We will prove Theorem 1.1 in the following form.

THEOREM 3.3. *If $\mathcal{F}$ is a finite collection of arcs of a circle with connected union, then the longest chains in $\mathcal{F}$ have a common arc.*

Proof. If the intersection graph of $\mathcal{F}$ is an interval graph, then Corollary 2.2 proves the claim. Otherwise, the cover $\mathcal{K}$ of the circle exists as defined above in the present section. Recall that every longest chain in $\mathcal{F}$ contains an arc of $\mathcal{K}$. Let $\mathcal{P}$ be a longest chain such that $|\mathcal{P} \cap \mathcal{K}|$ is the smallest possible. If $|\mathcal{P} \cap \mathcal{K}| = n$ then all longest chains contain each arc of $\mathcal{K}$. The theorem is also true in the case of $n = 1$ by the same reason. Thus we assume that $n \ge 2$ and $|\mathcal{P} \cap \mathcal{K}| < n$ or equivalently, $\mathcal{P} \cap \mathcal{K} = \{K_{a+1}, \ldots, K_{b-1}\} \neq \mathcal{K}$. We shall show that each longest chain contains K_{b-1}.

Assume otherwise, and let $\mathcal{Q}$ be a longest chain with $K_{b-1} \notin \mathcal{Q}$. By the choice of $\mathcal{P}$, we have $\mathcal{Q} \cap \mathcal{K} = \{K_{\ell+1}, \ldots, K_{m-1}\}$, $K_{b-1} \in \mathcal{P} \setminus \mathcal{Q}, K_{\ell+1} \in \mathcal{Q} \setminus \mathcal{P}$, and $K_b, K_{b+1}, \ldots, K_\ell \notin \mathcal{P} \cup \mathcal{Q}$. Let $\mathcal{R}$ be the chain $(K_b, \ldots, K_\ell)$, and let $\mathcal{R}$ be empty if $\ell = b - 1$.

Next we reorder $\mathcal{P}$ and $\mathcal{Q}$ as in Lemma 3.2, let $\mathcal{P}^* = \mathcal{P}_1 K_{b-1} \mathcal{P}_2$ and $\mathcal{Q}^* = \mathcal{Q}_1 K_{\ell+1} \mathcal{Q}_2$. Define the trails $\mathcal{C}_1 = \mathcal{P}_1 K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_1^r$ and $\mathcal{C}_2 = \mathcal{P}_2^r K_{b-1} \mathcal{R} K_{\ell+1} \mathcal{Q}_2$ where the superscript r indicates reversal of the sequence, and juxtaposition means the concatenation of arcs and subchains.

We claim that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are chains in $\mathcal{F}$. To verify this for $\mathcal{C}_1$, it is enough to show that $\mathcal{P}_1 \cap \mathcal{Q}_1 = \emptyset$. Assume on the contrary that $A \in \mathcal{P}_1 \cap \mathcal{Q}_1$. Then A preceeds $K_{\ell+1}$ in $\mathcal{Q}^*$, so that $A \subseteq K_\ell \cup K_{\ell+1}$ follows by Lemma 3.2. Observe that $A \not\subseteq K_\ell$, because $K_\ell \notin \mathcal{Q}$. Also $A \not\subseteq K_{\ell+1}$, because otherwise, $A \in \mathcal{P}$ would imply $K_{\ell+1} \in \mathcal{P}$, a contradiction. Therefore, we obtain

$$K_\ell \cap K_{\ell+1} \subseteq A \subseteq K_\ell \cup K_{\ell+1} \ .$$

Because A preceeds K_{b-1} in $\mathcal{P}^*$, by Lemma 3.2, we have

$$K_{b-1} \cap K_b \not\subseteq A \ .$$

The left hand sides of the two conditions on A are contradictory when $\ell = b - 1$. Thus $\mathcal{R}$ is non–empty, in particular, $K_\ell \notin \mathcal{P}$.

Because $A \in \mathcal{P}$ and $A \subseteq K_\ell \cup K_{\ell+1}$, either we have $(\operatorname{Supp} \mathcal{P}) \cap K_\ell \neq \emptyset$ which implies $K_\ell \in \mathcal{P}$ or we have $(\operatorname{Supp} \mathcal{P}) \cap K_{\ell+1} \neq \emptyset$ which implies $K_{\ell+1} \in \mathcal{P}$, each is impossible. Therefore A does not exist, hence $\mathcal{C}_1$ is a chain in $\mathcal{F}$.

The same type of argument applies and shows that $\mathcal{C}_2$ is a chain as well. Now we have $|\mathcal{C}_1| + |\mathcal{C}_2| \geq 2 + |\mathcal{P}| + |\mathcal{Q}|$ contradicting the fact that both $\mathcal{P}$ and $\mathcal{Q}$ are maximum. Thus $\mathcal{Q}$ does not exist, hence K_{b-1} is a common arc of all longest chains in $\mathcal{F}$. □

4. Remarks

4.1. Longest cycles

Gallai's question is also investigated for the longest cycles of a graph instead of its longest paths (see related problems in Voss [5]). The proof of Theorem 2.1 leads to a result parallel to Corollary 2.2 as follows.

THEOREM 4.1. *All maximum length cycles of a 2–connected interval graph have non–empty intersection.*

Proof. (Sketch) Note that 2–connectivity is required for the property that any two longest cycles share a common vertex. Cycles of an interval graph correspond to circular chains of intervals in its interval representation. The definition of the support of a circular chain $\mathcal{C} = (A_1, A_2, \ldots, A_t)$ will be the more symmetric expression

$$\operatorname{Supp}\mathcal{C} = (A_1 \cap A_2) \cup \ldots \cup (A_{t-1} \cap A_t) \cup (A_t \cap A_1) .$$

Making these changes, each step of the proof of Theorem 2.1 remains valid. □

The question whether the longest cycles have a non–empty intersection is open for further families of graphs, among others, for circular arc graphs.

4.2. Chordal graphs

So far we have not been able to determine whether the longest paths (or cycles) of every chordal graph have the Helly–property envisioned by Gallai. Klavžar and Petkovšek in [3] observed that the intersection of the longest paths is non–empty in a connected split graph (split graphs are chordal graphs whose complement is also chordal). In addition, Corollary 2.2 shows that Gallai's question has an affirmative answer for interval graphs, another distinguished subfamily of chordal graphs.

It is known that chordal graphs are exactly the intersection graphs of subtrees in some host tree. We could not extend our approach based on the interval representation of an interval graph to the subtrees in the tree representation of a chordal graph. However, using the subtree representation and the Helly–theorem on subtrees one can easily obtain that in a connected (or 2–connected) chordal graph there is a clique meeting every longest paths (or cycles). It is possible that all longest paths must go through a common vertex in that clique.

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