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# Packing Circuits into $K_N$.

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We shall pack circuits of arbitrary lengths into the complete graph $K_N$. More precisely, if $N$ is odd and $\sum_{i=1}^{t} m_i = \binom{N}{2}$, $m_i \geq 3$, then the edges of $K_N$ can be written as an edge disjoint union of circuits of lengths $m_1, \ldots, m_t$. Since the degrees of the vertices in any such packing must be even, this result cannot hold for even $N$. For $N$ even, we prove that if $\sum_{i=1}^{t} m_i \leq \binom{N}{2} - \frac{N}{2}$ then we can write some subgraph of $K_N$ as an edge disjoint union of circuits of lengths $m_1, \ldots, m_t$. In particular, $K_N$ minus a 1-factor can be written as a union of such circuits when $\sum_{i=1}^{t} m_i = \binom{N}{2} - \frac{N}{2}$. We shall also show that these results are best possible.

## 1. Introduction

The main result of this paper is the following:

**Theorem 1.** *Let $L = \sum_{i=1}^{t} m_i$, $m_i \geq 3$, with $L = \binom{N}{2}$ when $N$ is odd and $\binom{N}{2} - \frac{N}{2} - 2 \leq L \leq \binom{N}{2} - \frac{N}{2}$ when $N$ is even. Then we can write some subgraph of $K_N$ as an edge disjoint union of circuits of lengths $m_1, \ldots, m_t$.*

Note that we only guarantee the existence of circuits (even connected subgraphs) of the appropriate sizes. We have no control over their exact form, in particular we cannot ensure that they are all cycles.

**Corollary 2.** *Let $L = \sum_{i=1}^{t} m_i$, $m_i \geq 3$, then we can write some subgraph of $K_N$ as an edge disjoint union of circuits of lengths $m_1, \ldots, m_t$ if and only if either*

*1 $N$ is odd, $L = \binom{N}{2}$ or $L \leq \binom{N}{2} - 3$, or*
*2 $N$ is even, $L \leq \binom{N}{2} - \frac{N}{2}$.*

**Proof.** (Assuming Theorem 1).
If $L \leq \binom{N}{2} - 3$ ($N$ odd) or $L \leq \binom{N}{2} - \frac{N}{2} - 3$ ($N$ even) then add an extra $m_i$ so that $L = \binom{N}{2}$ or $L = \binom{N}{2} - \frac{N}{2}$ respectively. Using Theorem 1 and removing the final circuit proves the "if". For the "only if", assume we have such a packing. The vertices of the packed subgraph $G$ of $K_N$ must be of even degree, so the vertices of the edge complement

$G^c$ of this graph must have even degree if $N$ is odd and odd degree if $N$ is even. In the first case $G^c$ must have either zero or at least three edges. In the second case $G^c$ must have at least $\frac{N}{2}$ edges since each vertex must have degree at least one. □

A weaker result was proved in [1], which showed that for $\sum m_i \le \binom{N}{2}$ the circuits can be packed into a complete graph with about $9N/2$ vertices.

If we replace "circuits" by "cycles" in Theorem 1 and add the condition that all $m_i \le N$, then we obtain a conjecture of B. Alspach [4]. In general this conjecture is much stronger than Theorem 1, however since all circuits of length at most five are cycles, Theorem 1 does imply the main result of [7] which states that this conjecture holds when all the cycle lengths $m_i$ are three or five.

Theorem 1 has been proved by F. Hwang and S. Lin [9] in the special case when all the circuits are of the same length. More recently, the stronger Alspach's conjecture has also been proved when all the cycles are of equal length [5, 6].

Another application of Corollary 2 is in proving a conjecture of Burris and Schelp on vertex-distinguishing proper edge-colorings of graphs in the case of 2-regular graphs (see [1, 2] and especially [3] for details).

## 2. Notation

Write $K_n$ for a complete graph, $E_n$ for an empty graph and $C_n$ for a cycle on $n$ vertices. If we have a specific set $S$ of vertices in mind, we shall also use notations such as $K_S$ and $E_S$. Write $K'_n = K_n$ if $n$ is odd and $K'_n = K_n \setminus I$ if $n$ is even, where $I$ is any 1-factor of $K_n$. Here $G_1 \setminus G_2$ represents the graph $G_1$ with the edges of the subgraph $G_2$ removed. The graph $K'_n$ is the largest subgraph of $K_n$ for which each vertex has even degree and it has $\binom{n}{2}$ or $\binom{n}{2} - \frac{n}{2}$ edges depending on whether $n$ is odd or even. Write $P(v_1, v_2, \dots, v_r)$ for the *trail* of length $r-1$ on the vertices $v_i$ with edges $v_i v_{i+1}$. Also write $C(v_1, \dots, v_r)$ for the *circuit* $P(v_1, \dots, v_r, v_1)$ of length $r$. Note that we do *not* require the vertices $v_i$ to be distinct.

If $G_1$ and $G_2$ are vertex disjoint graphs, $G_1 \cup G_2$ will denote the union of $G_1$ and $G_2$. We also write $G_1 + G_2$ for the join of $G_1$ and $G_2$, i.e., the graph $G_1 \cup G_2$ with all edges connecting $G_1$ and $G_2$ included.

If $G_1$ and $G_2$ are graphs, a *packing* of $G_1$ into $G_2$ is a map $f\colon V(G_1) \to V(G_2)$ such that $xy \in E(G_1)$ implies $f(x)f(y) \in E(G_2)$ and the induced map on edges $xy \mapsto f(x)f(y)$ is an injection from $E(G_1)$ into $E(G_2)$. Note that $f$ is *not* required to be injective on vertices, so if $G_1$ contains a cycle or path, its image in $G_2$ will be a circuit or trail. A packing will be called *exact* if it induces a bijection between $E(G_1)$ and $E(G_2)$. We shall write $G_1 \mapsto G_2$ to mean that an exact packing of $G_1$ into $G_2$ exists. With this notation, the problem is one of packing a disjoint union of cycles $\cup_{i=1}^t C_{m_i}$ into $K_N$.

We shall write $G_1.G_2$ for the union of two graphs which are *not* disjoint on vertices. In other words, it is the image of a packing of $G_1 \cup G_2$ which is injective on $V(G_1)$ and injective on $V(G_2)$, but in which some vertices of $G_1$ are identified with some vertices of $G_2$. To make this precise, we shall always make it clear which vertices of $G_1$ are

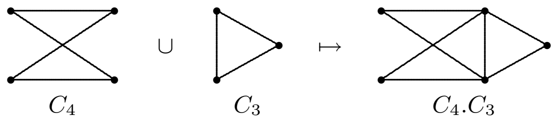



*Figure 1.* Example of linking graphs $C_4$ and $C_3$.

identified with which vertices in $G_2$. Vertices of $G_i$ that are identified will sometimes be called a *link* of $G_i$, and we shall call the identification a *linking* of $G_1$ and $G_2$.

Later on we shall define for some graphs *initial* and *final* links as (ordered) sets of vertices, possibly the same set. In these cases $G_1.G_2$ will identify the vertices of the final link of $G_1$ with the vertices of the initial link of $G_2$ (in the same order). Clearly this is only well defined if the two links are of the same size and no edge appears in both links. The initial link of the resultant graph will be that of $G_1$ and the final link will be that of $G_2$. This makes . an associative operation on graphs (when defined). Similarly, the initial link of $G_1 \cup G_2$ will be that of $G_1$ and the final link will be that of $G_2$. We shall write $G^{\cdot n}$ and $G^{\cup n}$ for $G.G \dots G$ and $G \cup \dots \cup G$ respectively, where in each case there are $n$ copies of $G$. We shall sometimes refer to $G^{\cdot n}$ (or the image of $G^{\cdot n}$ under some packing) as a *trail* of $G$'s.

Assume we have a graph of the form $C_a.C_b$ in which two cycles are linked at at least one vertex. We can pack $C_{a+b}$ into this graph by picking such a link vertex $v$ and going round $C_a$ starting at $v$, then going round $C_b$. By induction, if we have a sequence of cycles $C_{a_1}.C_{a_2} \dots C_{a_r}$ with each meeting the next in at least one vertex, we can pack a cycle of length $\sum_{i=1}^{r} a_i$ exactly into such a graph. We shall use this observation many times in what follows. We shall call a sequence of (connected) subgraphs in which each subgraph meets the next a *connected sequence* of subgraphs. A *circular connected sequence* will be a connected sequence in which the last subgraph also meets the first.

If we have an exact packing of a graph $G$ with triangles, then each edge in $G$ belongs to a unique triangle in the packing. We can define *trails*, *cycles*, etc., of triangles as trails, cycles, etc., (of edges) in which each edge belongs to a distinct triangle. The existence of a trail of triangles $T_1, \dots, T_r$ is stronger than the existence of a connected sequence of triangles. Indeed, for such a sequence to form a trail of triangles we need $V(T_i) \cap V(T_{i+1}) = \{v_i\}$ with $v_i \neq v_{i+1}$. Note that if we define initial and final links of a triangle as single *distinct* vertices, then this terminology fits with the previous definition of a trail of triangles as a packing of $T^{\cdot n}$ (see Figure 4).

Define graphs $G_{n,r}$, for $n$ and $r$ not both odd, to be $K'_{n+r}$ with the edges of a subgraph isomorphic to $K'_r$ removed. More explicitly, let $R$ be a set of $r$ vertices and $M$ a set of $n$ vertices and let $I_M$ (resp. $I_R$) be a 1-regular graph on $M$ (resp. $R$) when $n$ (resp. $r$) is even. Then

$$G_{n,r} = \begin{cases} K_M + E_R, & n \text{ even}, r \text{ odd}, \\ (K_M \setminus I_M) + E_R, & n \text{ even}, r \text{ even}, \\ K_M + I_R, & n \text{ odd}, r \text{ even}. \end{cases}$$

In all cases, we can write $K'_{n+r} = G_{n,r}.K'_r$ by identifying (linking) $R$ with the vertices

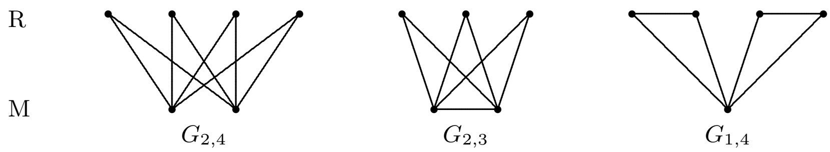



*Figure 2.* Examples of graphs $G_{n,r}$.

of $K'_r$, and identifying $I_R$ with the missing 1-factor of $K'_r$ when $r$ is even. Whenever we have a graph of the form $G_{n,r}$, the sets of vertices $M$ and $R$ will be defined as above.

Given a sequence of numbers $m_1, \dots, m_t$ with $m_i \geq 3$, write $L = \sum_{i=1}^{t} m_i$ for their sum. We shall define $n_j$ to be the number of terms $m_i$ that are equal to $j$, so that $L = \sum m_i = \sum j n_j$.

## 3. The Strategy

The proof of Theorem 1 will be by induction on $N$, so for many of the lemmas we shall assume the following:

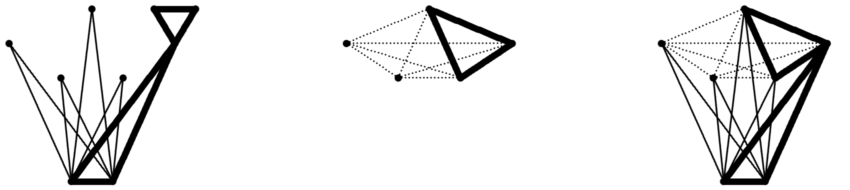
**Induction Hypothesis.** The conditions of Theorem 1 hold for $N$ and Theorem 1 holds for all smaller values of $N$.

The idea behind the proof is to pack some cycles into $K_N \setminus K_r$ and the rest by induction into $K_r$ to obtain a packing of all the cycles into $K_N$. For reasons of parity, we use the $G_{n,r}$ defined above with $n + r = N$ instead of $K_N \setminus K_r$ since every vertex of $G_{n,r}$ has even degree. We can pack $G_{n,r} \cup K_r$ (or $G_{n,r} \cup K'_r$ if $n$ is odd) into $K_N$. The advantage of this approach is that we can choose the subset of cycles that we pack into $G_{n,r}$, leaving any "awkward" cycles to be packed by induction into $K_r$.

One major complication is that an exact packing into $G_{n,r}$ may not exist. Indeed, there may be no subset $S \subseteq \{1, \dots, t\}$ with $\sum_{i \in S} m_i = |E(G_{n,r})|$. To allow greater flexibility, we need to allow some "overflow" out of $G_{n,r}$. To be precise, we shall pack our subset of cycles $\cup_{i \in S} C_{m_i}$ into graphs of the form $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$, where we have linked some extra cycles $C_{l_1}, \dots, C_{l_k}$ to $G_{n,r}$ at some vertices of $R$. The extra cycles and the manner in which they are linked to $G_{n,r}$ are chosen so as to make such a packing possible.

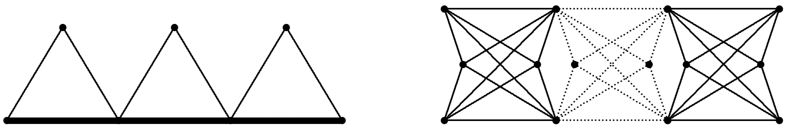
We shall then pack $(\cup_{i=1}^{k} C_{l_i}) \cup (\cup_{i \in S^c} C_{m_i})$—the $C_{l_i}$ and the remaining $C_{m_i}$—into $K_r$ using the Induction Hypothesis. We aim to combine these two packings into one packing into $G_{n,r}.K_r \subseteq K_N$ (or $G_{n,r}.K'_r = K_N$ when $n$ is odd). However, when packing into $K_N$, we must make sure that the packed cycles $C_{l_i}$ in $K_r$ meet $G_{n,r}$ at the correct vertices so that we get a packing of $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$ and $\cup_{i \in S^c} C_{m_i}$ into $K_N$ (see Figure 3) Lemma 3 of Section 4 gives some sufficient conditions that ensure that this is possible. The details are somewhat technical, but the basic idea is that the strategy works provided that there are not too many additional cycles $C_{l_i}$ and they are not linked to $G_{n,r}$ at too many vertices.

For small values of $n$ we can write $G_{n,r}$ as a connected sequence of squares and triangles. This and Lemma 3 allows us to prove some special cases of Theorem 1 when we either



$C_8 \cup C_6 \mapsto G_{2,5}.C_3$ and $C_3 \cup C_7 \mapsto K_5$ $\Rightarrow$ $C_8 \cup C_6 \cup C_7 \mapsto K_7$

*Figure 3.* Example with $m_1 = 8$, $m_2 = 6$, $m_3 = 7$, $l_1 = 3$, $S = \{1, 2\}$.



$T^{\cdot 3} = T.T.T$ $O^{\cdot 3} = O.O.O$

*Figure 4.* A trail of Triangles and a trail of Octahedra.

have a large number of cycles $C_{m_i}$ of certain lengths (Section 4, Lemmas 4 and 5) or if $N$ is sufficiently small (Section 5). These results are needed to cover some special cases not handled by the general methods of later sections. However, for more general combinations of cycle lengths this approach will fail since we would need to link too many additional cycles $C_{l_i}$ to $G_{n,r}$. Instead, for more general combinations of cycle lengths we need to look at $G_{n,r}$ for large $n$. For these $n$, $G_{n,r}$ becomes a much more complicated graph, so we shall decompose it into simpler graphs.

Our strategy then is to decompose some $G_{n,r}$ with $n+r = N$ into connected sequences of octahedra ($K_6'$). We shall then devise methods of packing our cycles $C_{m_i}$ into such connected sequences. Octahedra were chosen because they are large enough to allow a wide range of packings by cycles but sufficiently small to analyze these packings. The situation is complicated by the fact that in general $G_{n,r}$ cannot be packed with octahedra exactly. Indeed, we will only be able to get such packings when $N \equiv 2 \bmod 4$. For other $N$, we shall need to pack some other small graphs into $G_{n,r}$ to fill the "gaps" left by the octahedra.

Sections 6 and 7 deal with the decomposition of $G_{n,r}$ into octahedra. In Section 6 we pack $G_{n,r}$ for suitable $n$ and $r$ with trails of triangles. In Section 7, we double up vertices to obtain packing of some larger $G_{n,r}$ with trails of octahedra together will some other small graphs. Generating octahedra from trails of triangles ensures that the octahedra are linked together in a suitable manner.

Define $O$ to be the graph of an octahedron, so $O = K_6' = K_{2,2,2} = E_2 + E_2 + E_2$. The first $E_2$ will be the initial link and the last $E_2$ will be the final link of $O$. In fact by symmetry it does not matter which $E_2$'s are chosen, or the order of the vertices in either link. The trail of octahedra generated in Section 7 is then a packing of $O^{\cdot r} = O.O\ldots O$ where the final link of each octahedron is linked to the initial link of the next one (recall the notation defined in Section 2 and see Figure 4).

Having packed trails of octahedra into some $G_{n,r}$, we now need to pack our cycles $C_{m_i}$ into these trails. In general, it will not be possible to group the cycles into combinations with total length $|E(O)| = 12$. We therefore need to allow some "overflow" from one octahedra to the next. This is done in Section 8, where we define the "overflow" graphs $L_n$ and inductively pack (almost) arbitrary combinations of cycles into trails of octahedra. The proof of Theorem 1 will follow, at least in the case when $N \equiv 2 \bmod 4$. For other $N$, $G_{n,r}$ is not packed completely with octahedra, and we need to pack cycles into some other small graphs. In the case $N \equiv 3 \bmod 4$ these extra graphs are triangles. We can pair the triangles up with octahedra into trails of the form $T.O.T.O\ldots$, and we can handle $T.O$ in very much the same way as $O$. The cases $N \equiv 2$ and 3 mod 4 are proved in Section 9. For $N \equiv 1 \bmod 4$, the extra graphs are $K_5$'s linked at a single vertex. These can also be packed with a wide variety of cycles. The proof of Theorem 1 in this case is given in Section 10. Since $K_5$'s cannot pack every cycle length, we need to handle a number of special cases separately. These are the special cases in Section 4 described above. The most difficult case is when $N \equiv 0 \bmod 4$. For these we pack cycles into linked $G_{4,4}$'s. The graph $G_{4,4}$ has 20 edges, which makes packing cycles into $G_{4,4}$ much more complicated than octahedra and $K_5$'s. Also, $G_{4,4}$'s cannot be used to pack triangles, so we run into problems when there are many triangles to be packed. In particular, we need to construct several special packings to handle the case when $N$ is small and there are many triangles. Section 11 gives the details of the proof in this case.

## 4. Some Packing Lemmas

As described in Section 3, we shall pack some subset $\cup_{i\in S} C_{m_i}$, $S \subseteq \{1,\ldots,t\}$, of the cycles into a graph of the form $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$ and then pack the $C_{l_i}$ and the remaining $C_{m_i}$ into $K_r$ using the Induction Hypothesis. We aim to combine these two packings into one packing into $G_{n,r}.K_r \subseteq K_N$ (or $G_{n,r}.K'_r = K_N$ if $n$ is odd). However, when packing into $K_N$, we must make sure that the packed cycles $C_{l_i}$ in $K_r$ meet $G_{n,r}$ at the correct vertices so that we get a packing of $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$ and $\cup_{i\in S^c} C_{m_i}$ into $K_N$. The following lemma gives some sufficient conditions that ensure that this is possible.

**Lemma 3.** *Assume the Induction Hypothesis. Suppose that we can pack some subset of the cycles $\cup_{i\in S} C_{m_i}$ exactly into some graph of the form $G_{n,r}.(\cup_{i=1}^{k} C_{l_i})$ where $N = n + r$ and assume the links $V(C_{l_i}) \cap V(G_{n,r})$ are pairwise disjoint subsets of $R$ (of $G_{n,r}$). Then we can pack all the cycles into some subgraph of $K_N$ provided any one of the following conditions holds.*

1. *$n$ is even, $k \le 9$ and $|V(C_{l_i}) \cap R| = 1$,*
2. *$n$ is odd, $k \le 3$ and $V(C_{l_i}) \cap R = \{v_i\}$ where $v_i$ are in distinct components of $I_R$,*
3. *$n$ is even, $|V(C_{l_i}) \cap R| = 1$ for $i \ge 2$, $l_1 = 3$, 4 or 5 and $k \le 5$, 2 or 1 respectively,*
4. *$n$ is even, $k \le 6$, $|V(C_{l_1}) \cap R| = 2$ and $|V(C_{l_i}) \cap R| = 1$ for $i \ge 2$,*
5. *$n$ is even, $k = 2$, $l_1 = 3$ and $|V(C_{l_i}) \cap R| = 2$ for $i = 1, 2$.*

**Proof.** By the Induction Hypothesis, we can find a packing $f$ which packs the remaining cycles $\cup_{\in S^c} C_{m_i}$ and all the $C_{l_i}$ into $K_r$. The conditions on the total length of these cycles in Theorem 1 are satisfied since $|E(K'_N)| - |E(G_{n,r})| = |E(K'_r)|$ and $N$ even implies $r$ even. We shall consider the five cases separately.

1. If we take $s \geq 1$ of the circuits $f(C_{l_i})$, then they contain in total at least $3s$ edges of $K_r$. Assume these circuits meet no more than $s-1$ vertices. By looking at the degrees of these vertices we see that $(s-1)\lfloor\frac{1}{2}(s-2)\rfloor \geq 3s$. This is not possible when $s \leq k \leq 9$, so the circuits meet at least $s$ vertices. By Halls' marriage theorem, we can pick distinct vertices $u_1, \dots, u_k$ with $f(C_{l_i})$ meeting $u_i$. By a suitable identification of the vertices of $K_r$ with those of $R$, we can assume that $V(C_{l_i}) \cap R = \{u_i\}$. We now have a packing of $(G_{n,r}.(\cup_{i=1}^k C_{l_i})) \cup (\cup_{i\in S^c} C_{m_i})$ into some subgraph of $G_{n,r}.K_r \subseteq K_N$ by extending $f$ by the identity on $G_{n,r}$. Composing with the given packing of $(\cup_{i\in S} C_{m_i})$ into $(G_{n,r}.(\cup_{i=1}^k C_{l_i}))$ gives the result.

2. Let $G$ be the image of $f$ in $K_r$. Now $r$ is even and $N$ is odd, so $|E(G)| = \binom{N}{2} - |E(G_{n,r})| = \binom{r}{2} - \frac{r}{2}$. Therefore the complementary graph $I = G^c$ must have $r/2$ edges and all vertices of odd degree. Hence $I$ is a 1-factor of $K_r$ and $G = K_r \setminus I$. Since $(2s-2)\lfloor\frac{1}{2}(2s-3)\rfloor < 3s$ for $s \leq k \leq 3$, each set of $s$ circuits $f(C_{l_i})$ meets at least $2s-1$ vertices and hence at least $s$ components of $I$. Pick $u_1, \dots, u_k$ in distinct components of $I$ with $f(C_{l_i})$ meeting $u_i$. We can now identify $G$ with $K_R \setminus I_R$ in such a way that $I$ corresponds to $I_R$ and $u_i = v_i$. The result now follows as in part 1.

3. The image $f(C_{l_1})$ must be a cycle since $l_1 \leq 5$. We can therefore match the vertices of $V(C_{l_1}) \cap R$ with their images under $f$ in $K_r$. To match the remaining vertices, we need to find distinct vertices $u_2, \dots, u_k$ in $K_r$ with $f(C_{l_i})$ meeting $u_i$ and $u_i \notin f(C_{l_1})$. It is sufficient to ensure that $s \leq k-1$ of the other circuits meet at least $s + l_1$ vertices given that a $l_1$-cycle already uses $l_1$ edges. This can be checked as above by showing $(s + l_1 - 1)\lfloor\frac{1}{2}(s + l_1 - 1)\rfloor < 3s + l_1$ holds for $1 \leq s \leq k-1$ when $l_1 = 3, 4, 5$ and $k \leq 5, 2, 1$ respectively.

4. We must first show that if we pick two vertices $w_i$ and $w_j$ of $C_{l_1} = C(w_1, \dots, w_{l_1})$, we can ensure their images under $f$ are distinct. Assume this is not the case and the image under $f$ is $u = f(w_i) = f(w_j)$. Modify $f$ on $C_{l_1}$ by cyclically moving its image vertices one step around the circuit, say $f_+(w_i) = f(w_{i+1})$ and $f_+ = f$ on all other cycles. If the images of $w_i$ and $w_j$ are still identical, say $u' = f_+(w_i) = f_+(w_j)$, then the edge $uu'$ occurs twice in the image $f(C_{l_1})$, contradicting the definition of the packing $f$. The argument now proceeds as before. We check that any $s \leq k-1 \leq 5$ of the remaining circuits meet at least $s+2$ vertices (so the matching of vertices to the remaining circuits avoids the two vertices $f(w_i)$ and $f(w_j)$).

5. It is enough to require that $|f(C_{l_2} \cap R)| = 2$ and $|f(C_{l_2} \cap R) \cap f(C_{l_1})| \leq 1$ since by cyclically permuting $f$ on $C_{l_1}$ we can then make $f(C_{l_1} \cap R)$ and $f(C_{l_2} \cap R)$ disjoint pairs of vertices. Let $f(C_{l_1}) = \{u_1, u_2, u_3\}$ and $C_{l_2} \cap R = \{w_1, w_2\}$. Since $f(C_{l_2}) \not\subseteq \{u_1, u_2, u_3\}$ we can cyclically permute $f$ on $C_{l_2}$ so that $f(w_1) \notin \{u_1, u_2, u_3\}$. If $f(w_2) \neq f(w_1)$ then we are done. Otherwise let $u = f(w_1) = f(w_2)$ and cyclically permute $f$ one step forward and backwards on $C_{l_2}$ as in part 4 to get $f_+$ and $f_-$ with $f_+(w_1) \neq f_+(w_2)$ and $f_-(w_1) \neq f_-(w_2)$. Since $f(w_1) = f(w_2)$ the points $w_1$ and $w_2$ must be distance at least three apart on $C_{l_2}$. Now if $f_+(w_1) = f_-(w_2) = u'$ then we have two edges in $C_{l_2}$ whose image under $f$ is $uu'$ contradicting the definition of the packing $f$. Similarly we can't

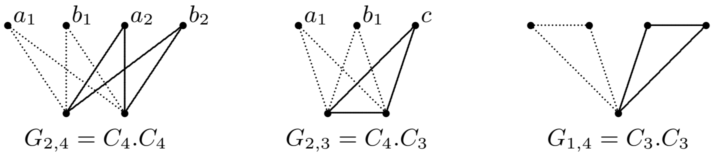



*Figure 5.* Packing $G_{1,r}$ and $G_{2,r}$ with $C_3$'s and $C_4$'s.

have $f_+(w_2) = f_-(w_1)$. Hence, $f_\pm(w_1), f_\pm(w_2)$ are four distinct vertices and so at least one is not in $\{u_1, u_2, u_3\}$. Replacing $f$ by either $f_+$ or $f_-$ then gives the result. □

The next two lemmas use Lemma 3 with small values of $n$ to obtain packings in some special cases when we have enough cycles of lengths divisible by three or four or equal to seven. Recall that $n_j$ is the number of cycles of length $j$.

**Lemma 4.** *Assume the Induction Hypothesis. If $\sum jn_{4j} \geq \frac{1}{2}(N-3)$ then the conclusion of Theorem 1 holds.*

**Proof.** The cases when $N \leq 3$ are trivial, so assume $N \geq 4$. Let $n = 2$, $r = N-2$ and consider the graph $G_{2,r}$. Write $R$ as the union of pairs $\{a_i, b_i\}$, $i = 1, \dots, \lfloor r/2 \rfloor$ and an extra vertex $\{c\}$ if $r$ is odd (see Figure 5). Then $S_i = E_M + E_{\{a_i,b_i\}} = E_2 + E_2$ are squares. Since the squares are all connected (via $M$), these form a connected sequence of squares $S_1.S_2 \dots S_{\lfloor r/2 \rfloor}$. Take each cycle $C_{m_i}$ with $m_i \equiv 0 \bmod 4$ in turn and pack them into $m_i/4$ consecutive squares in this sequence. We continue until we run out of squares. To pack the last such $C_{m_i}$ we may need to link an extra cycle to the vertex $a_{\lfloor r/2 \rfloor}$, say, to pack all of this cycle exactly. Since the total length of all such $C_{m_i}$ is at least $4\lfloor r/2 \rfloor$, we shall not run out of these cycles. If $N$ is even, we have used all the edges of $G_{2,r} = E_M + E_R$ and we are done by part 1 of Lemma 3 with $k \leq 1$.

Now assume $N$ is odd. The remaining edges of $G_{2,r} = K_M + E_R$ form the triangle $T = K_M + E_{\{c\}} = K_2 + E_1$. If one of the remaining cycles $C_{m_i}$, $m_i \not\equiv 0 \bmod 4$, is of length 3 or $\geq 6$, we can pack this into $T$ (linking a cycle $C_{m_i-3}$ to vertex $c$ if $m_i \geq 6$). By part 1 of Lemma 3 with $k \leq 2$ we are done in this case. This also covers the case when $\sum jn_{4j} \geq \frac{1}{2}(N+1)$ and not all of these cycles are squares. (Link a $C_5$ to $c$, and start by packing a cycle of length $m_i \geq 8$, $m_i \equiv 0 \bmod 4$, into this $C_5$, $T$ and possibly some $S_i$ first. Then pack the remaining $C_{m_i}$ with $m_i \equiv 0 \bmod 4$ into the other $S_i$'s as above.)

Now assume there are two cycles of length 5. We can pack these into $K_{M\cup\{a_1,b_1,c\}} = K_5$ (as $C(a_1, b_1, c, u_1, u_2)$ and $C(a_1, c, u_2, b_1, u_1)$ where $M = \{u_1, u_2\}$). Now pack the the cycles of length $m_i \equiv 0 \bmod 4$ into $S_2, \dots, S_{\lfloor r/2 \rfloor}$ as above. This uses all the edges of $G_{2,r}$, the triangle $C(a_1, b_1, c)$ and possibly one other cycle linked to $a_{\lfloor r/2 \rfloor}$, say. We are now done by part 3 of Lemma 3 with $l_1 = 3$, $k \leq 2$.

The only remaining cases are when every cycle has length divisible by 4 except possibly for a single $C_5$. In this case $L = \binom{N}{2} \equiv 0$ or $1 \bmod 4$, so $N \geq 7$ and $\sum jn_{4j} \geq \frac{1}{4}(\binom{N}{2} - 5) \geq \frac{1}{2}(N+1)$. We can therefore assume we have at least $\frac{1}{2}(N+1)$ squares. Pack $\frac{1}{2}(N-5)$ squares into $S_2, \dots, S_{\lfloor r/2 \rfloor}$, and 3 squares as $C(a_1, u_1, u_2, b_1)$, $C(a_1, u_2, c, a_2)$ and

$C(b_1, u_1, c, b_2)$. This gives a packing using the edges of $G_{2,r}$ and $C(a_1, b_1, b_2, c, a_2) = C_5$. We are now done by part 3 of Lemma 3 with $l_1 = 5$, $k = 1$. □

**Lemma 5.** *Assume the Induction Hypothesis with $N$ odd. If $\sum jn_{3j} \geq \frac{1}{2}(N-1)$ or $n_7 \geq \frac{1}{2}(N-1)$ then the conclusion of Theorem 1 holds.*

**Proof.** First assume $\sum jn_{3j} \geq \frac{1}{2}(N-1)$ and let $n = 1$, $r = N-1$. Then $G_{1,r} = K_1 + I_R$ forms $r/2$ triangles linked by a common vertex. Pack cycles of length divisible by 3 into this graph in a manner analogous to Lemma 4. We may need to link a cycle to one of the vertices of $R$ to pack the last cycle. Using part 2 of Lemma 3 with $k \leq 1$ gives the result. Now assume $n_7 \geq \frac{1}{2}(N-1)$. Let $n = 3$, $r = N-3$. Then $G_{3,r} = K_3 + I_R$ is an edge disjoint union of $r/2 - 1$ graphs $E_3 + K_2$, each of which can be packed with a single $C_7$, and one $K_3 + K_2 = K_5$, which can be packed with a $C_7$ and a $C_3$. By permuting the vertices of this $K_5$, we can ensure that the $C_3$ meets $R$, so by linking a $C_4$ to one of the vertices of $R$ we can pack another $C_7$. We now have a packing of $r/2 + 1 = \frac{1}{2}(N-1)$ $C_7$'s into $G_{3,r}.C_4$. The result now follows from part 2 of Lemma 3 with $k = 1$. □

## 5. Packing $K_N$ for small $N$.

Since the methods that we shall use in subsequent sections do not always apply for small $N$, it will be necessary to give alternative proofs for these $N$. We include these proofs first, since some of these results will be useful later. The cases treated here require long and tedious case by case checking. As a result, we do not include all the details. Since the methods used in these proofs are not used again (except briefly at the end of Section 11), the reader may wish to skip the proofs in this section on first reading. For $N \leq 10$, most of the cases are also implied by the results in [11].

**Lemma 6.** *Assume the Induction Hypothesis with $N \leq 11$, $N$ odd. Then the conclusion of Theorem 1 holds.*

**Proof.** The cases when $N < 5$ are clear, so we shall assume $5 \leq N \leq 11$. Write each cycle length $m_i \neq 5$ as a sum of 3's and 4's. For example, $11 = 4 + 4 + 3$ and $6 = 3+3$. All cycle lengths except for 5 can be written in this way (though not necessarily uniquely). Let $t_T$, $s_T$ and $p_T$ be the total number of 3's, 4's and remaining $C_5$'s so that $L = \binom{N}{2} = 3t_T + 4s_T + 5p_T$. The idea is to pack some of the cycles as connected sequences of triangles, squares and pentagons in certain explicit constructions.

Assume first that $s_T \geq \frac{1}{2}(N-3)$. If all of the 4's occur in cycles that are written without 3's, then these cycles have lengths divisible by four and we are done by Lemma 4. Hence we may assume that at least one of the cycles, $C_{m_1}$ say, is written with both 3's and 4's. Using the construction of Lemma 4 we can write $G_{2,N-2}$ as a connected sequence $T.S_1 \ldots S_{\lfloor r/2 \rfloor}$, where $T$ is the triangle $K_M + E_{\{c\}}$. Now pack the cycle $m_1 = 3 + r.4 + s.3$ into the triangle and the first $r$ squares. Link, if necessary, a $C_{3s}$ to the $r$th square at a vertex of $R$. Proceed with each other cycle in turn that has at least one 4 in its expansion, packing at least one square and linking a cycle to a vertex of $R$ whenever necessary. Since $s_T \geq \frac{1}{2}(N-3)$, we shall pack the whole of $G_{2,r}.(\cup C_{l_i})$ where the $C_{l_i}$ are linked to $G_{2,r}$ at single distinct vertices of $R$. The result now follows from part 1 of Lemma 3 with $k \leq \frac{1}{2}(N-3) \leq 4$.

Now assume $t_T \geq \frac{1}{2}(N-1)$. Using the first construction of Lemma 5, we can write $G_{1,N-1}$ as a connected sequence of triangles. Pack cycles written with 3's into this connected sequence linking extra cycles $C_{l_i}$ when necessary at vertices in $R$. We pack the cycles written with 3's only first. Hence if we need to link more than one cycle to $G_{1,N-1}$, each linked cycle will arise by packing a $C_{m_i}$ with $m_i$ written with at least one "4". Since we can assume $s_T \leq \frac{1}{2}(N-3) - 1 \leq 3$, we shall need to link at most three cycles $C_{l_i}$ to $G_{1,N-1}$. The cycles $C_{l_i}$ are linked at single vertices to distinct components of $I_R$, so we are done by part 2 of Lemma 3.

Finally, assume $p_T \geq \frac{1}{2}(N-1)$. Remove $\frac{1}{2}(N-1)$ $C_5$'s, add a $C_{(N+1)/2}$ and use the Induction Hypothesis to pack these cycles into $K_{N-2}$. (The total number of edges is $\binom{N}{2} - \frac{5}{2}(N-1) + \frac{1}{2}(N+1) = \binom{N-2}{2}$.) By suitably labeling vertices of $K_{N-2} = K_R$ as $a_i$, $1 \leq i \leq \frac{1}{2}(N-1)$ and $b_i$, $1 \leq i \leq \frac{1}{2}(N-3)$, we can assume the cycle $C_{(N+1)/2}$ is packed as $C(x, a_1, a_2, \ldots, a_{(N-1)/2})$ where $x = b_1$ when the cycle is packed as a cycle, and $x = a_3$ in the special case when $N = 11$ and $C_{(N+1)/2} = C_6$ is packed as two connected triangles. Now pack $\frac{1}{2}(N-1)$ pentagons into $G_{2,N-2}.C_{(N+1)/2}$ as follows:

$$C(a_i, u_1, b_i, u_2, a_{i+1}), \quad 1 \leq i \leq \tfrac{1}{2}(N-3), \qquad \text{and} \qquad C(a_1, u_2, u_1, a_{(N-1)/2}, x),$$

where $M = \{u_1, u_2\}$. This uses the edges of $C_{(N+1)/2}$ and $G_{2,N-2}$ and hence gives a packing of the original cycles into $G_{2,N-2}.K_{N-2} = K_N$.

If none of these conditions hold, the total number of edges can be at most

$$3(\tfrac{1}{2}(N-1)-1) + 4(\tfrac{1}{2}(N-3)-1) + 5(\tfrac{1}{2}(N-1)-1) = 6N-22.$$

However $L = \binom{N}{2} > 6N - 22$ when $N \geq 5$. □

Packing $K_N$ for even $N$ is more difficult since we cannot use Lemma 5 in this case. Before we prove the result for small even $N$, we shall list some explicit packings that we will need. Recall that a circular connected sequence of triangles is an edge-disjoint set of triangles $T_1, \ldots, T_r$ with $V(T_i) \cap V(T_{i+1}) \neq \emptyset$ and $V(T_r) \cap V(T_1) \neq \emptyset$.

**Lemma 7.**

1 *For $N = 8, 10$ and $4 \leq m \leq N$, or for $N = 14$ and $7 \leq m \leq N$ we can pack $K_N$ with a circular connected sequence of triangles $T_1, \ldots, T_r$ and a cycle $C_m$ with $3r + m \geq \binom{N}{2} - \frac{N}{2} - 2$ and the cycle packed as a cycle (i.e., the packing is injective on the vertices of $C_m$).*

2 *We can pack $G_{8,4}$ exactly with a circular connected sequence of triangles together with either one $C_5$ meeting $R$ or two linked $C_4$'s which meet distinct vertices of $R$ (of $G_{8,4}$).*

3 *We can pack $C_3 \cup C_4 \cup C_5$, $C_6 \cup C_6$ and $C_6 \cup C_3 \cup C_3$ into $K'_6$ with the $C_6$'s packed as cycles.*

4 *We can pack $C_m$ and two connected triangles into $K_6$ when $m = 4$ or $5$.*

**Proof.** (We made use of a computer search to find the constructions in this proof.) Label the vertices of $K_N$ as 0 to $N-1$ in part 1, and the vertices of $G_{8,4} = (K_M \setminus I_M) + E_R$ as $R = \{0, 1, 2, 3\}$, $M = \{4, 5, 6, 7, 4', 5', 6', 7'\}$ in part 2 where $I_M$ consists of the edges $nn'$. Remove the following circular connected sequences of triangles from $K_8$, $K_{10}$, $K_{14}$ and $G_{8,4}$ respectively

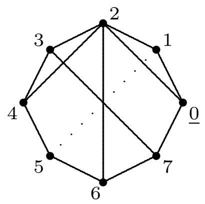
From $K_8$ $C(\underline{0},3,5), C(\underline{0},4,6), C(1,3,6), C(1,4,7), C(2,5,7)$

From $K_{10}$ $C(\underline{0},3,6), C(\underline{0},4,7), C(0,5,\underline{8}), C(1,4,\underline{8}), C(1,3,7)$
$C(1,6,9), C(2,4,9), C(2,6,8), C(2,\underline{5},7), C(3,\underline{5},9)$

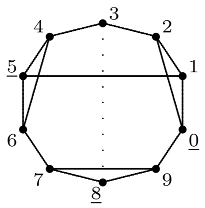
From $K_{14}$ $C(\underline{0},4,6), C(\underline{0},5,7), C(0,8,10), C(0,\underline{9},11), C(1,6,\underline{9}), C(1,3,5)$
$C(1,4,8), C(1,7,11), C(1,10,12), C(2,5,10), C(2,4,9), C(2,6,11)$
$C(2,7,13), C(2,8,12), C(3,6,12), C(3,7,10), C(3,8,11), C(3,9,13)$
$C(4,10,13), C(4,7,12), C(5,9,12), C(5,11,13), C(6,8,13)$

From $G_{8,4}$ $C(0,6',7'), C(0,4',5'), C(0,6,\underline{7}), C(1,5',\underline{7}), C(1,5,7'), C(1,4',6')$
$C(2,4',5), C(2,4',7), C(2,5',7'), C(2,4,6), C(4,5',6'), C(3,6',7)$
$C(3,5',6), C(3,4',5), C(3,4,7'), C(4',6,7')$

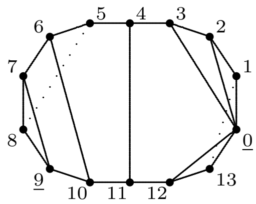
The resulting graphs are shown below (where we have removed some unimportant vertices and edges for clarity).



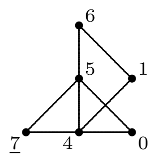
$H_8$



$H_{10}$



$H_{14}$



$H_{8,4}$

We shall pack $H_8$, $H_{10}$ and $H_{14}$ with $C_m$ and some triangles. These triangles will meet the underlined vertices above so can be inserted in the circular connected sequences at the points where the sequences are connected via these vertices. The result follows by inspection for $N = 8$, $m = 8, 7$ (with no triangles), $m = 6, 5, 4$ (with triangle $C(0, 1, 2)$), $N = 10$, $m = 10, 9, 8$ (with no triangles), $m = 7, 6, 5$ (with triangle $C(7, 8, 9)$), $N = 14$, $m = 14, 13$ (with no triangles), $m = 12, 11, 10$ (with triangle $C(0, 1, 2)$), $m = 9$ (with triangles $C(0, 1, 2)$ and $C(7, 8, 9)$) and $m = 8, 7$ (with triangles $C(0, 1, 2)$ and $C(0, 12, 13)$). For $N = 10$, $m = 4$, delete $C(2, 4, 9)$ from the connected sequence of triangles (leaving it still connected), use the three triangles in $H_{10}$ and the square $C(2, 3, 4, 9)$. Similarly $H_{8,4}$ can be packed with two squares or a pentagon and a triangle $C(7, 4, 5)$. Parts 1 and 2 now follow. Parts 3 and 4 are easy to check. □

Following the proof of Lemma 6, we shall write the cycle lengths $m_i$ as a sum of 3's, 4's and 5's. However, unlike Lemma 6, we shall write them so as to use as many 4's as possible. For example, $9 = 4 + 5$, $13 = 4 + 4 + 5$, $10 = 4 + 3 + 3$. It is clear that all $m_i$ will be written in one of the following forms:

$$3, \quad 5, \quad 3+3, \quad n.4, \quad n.4+3, \quad n.4+5, \quad \text{or} \quad n.4+3+3 \quad (n \geq 1).$$

As in Lemma 6, we write $t_T$, $s_T$ and $p_T$ for the total number of 3's, 4's and 5's so that $L = 3t_T + 4s_T + 5p_T$. As before, we shall attempt to pack some of the cycles as connected sequences of triangles, squares and pentagons.

**Lemma 8.** *Assume the Induction Hypothesis with $N$ even, $N \leq 12$ or with $N = 16$ and $2s_T + p_T \leq 7$. Then the conclusion of Theorem 1 holds.*

**Proof.** The cases when $N < 6$ are clear, so assume $N \geq 6$. Let $n = 2$, $r = N - 2$ and consider packings of $G_{2,r}$. As in Lemma 4 this can be written as a connected sequence of $r/2$ squares. If $s_T \geq r/2$, we are done by a similar (but slightly simpler) argument to that in Lemma 6, so assume $s_T < r/2$. We shall use the following packings of $t$ triangles, $p$ pentagons and $s$ squares into $G_{2,r}.C_l$ where $t$ is even, $l = t + p \geq 3$ and $t + 2p + 2s = r$. Label the vertices of $R$ as $a_1, \dots, a_l, a_{l+1} = a_1$ and $b_1, \dots, b_{r-l}$ and let $M = \{u_1, u_2\}$. Pack pairs of triangles as $C(a_i, u_1, a_{i+1})$ and $C(a_{i+1}, u_2, a_{i+2})$ with $i$ odd, $1 \leq i < t$. Then pack pentagons as $C(a_i, u_1, b_j, u_2, a_{i+1})$ for $1 \leq j \leq p$, $i = j + t$. Finally, pack squares as $C(b_j, u_1, b_{j+1}, u_2)$ with $j > p$. This uses all the edges of $G_{2,r}$ and the cycle $C_l = C(a_1, \dots, a_l)$. All the squares and pentagons are connected to every other cycle in this packing (via $M$) and the triangles form a connected sequence. We shall consider four cases.

1. The cases $r/2 - 2 \leq s_T < r/2$, $1 \leq p_T \leq 2$, $6 \leq N \leq 12$.

Use the construction above with $s = r/2 - 2$, $p = 1$, $t = 2$, $l = 3$. Pack cycles written with 4's or 5's into the squares and pentagons of this construction. We may need to link extra cycles to the squares and pentagons in this construction to pack these cycles fully. At most $s + p \leq 4$ extra cycles will be needed and they can be linked to distinct vertices $b_j$ (so avoiding $C_l$). If these cycles also use at least two 3's between them, we can use them to pack the remaining two triangles in the construction. Otherwise use two cycles $C_{m_i}$ of length 3 or one cycle of length 6 (if they exist). By considering the total number of edges it can be shown that we can fill the two triangles in the construction except in three cases. The first is packing $C_7 \cup C_7 \cup C_5 \cup C_5$ into $K_8$. For this split one of the $C_7$'s into a $C_3$ and a $C_4$. Pack these and check that we can make the $C_3$ meet the $C_4$ in the final packing. The next case is $t_T = s_T = p_T = 1$ in $K_6$. This case follows from part 3 of Lemma 7 (the cycles clearly all meet in this packing since there are only six vertices). Finally there is the case $p_T = 2$, $t_T = s_T = 0$ in $K_6$. In this case we have a packing into $K_5$ which is a subgraph of $K_6$. In the general case we are done by part 3 of Lemma 3 with $l_1 = l = 3$, $k \leq 4$.

2. The cases $r/2 - 2 \leq s_T < r/2$, $p_T = 0$, $6 \leq N \leq 12$.

Use the construction above with $s = r/2 - 2$, $p = 0$, $l = t = 4$. Pick a minimal subset of cycles that use at least $s$ 4's between them and as many 3's as possible. By minimality, these cycles must use at most $k$ 3's where $k \leq 2s \leq 6$. By counting edges, it can be shown that if $k < 4$ then there are enough cycles of lengths 3 and 6 to pack all the remaining $4 - k$ triangles of the construction except in the case $s_T = 1$, $t_T = 2$ in $K_6$. This exceptional case can be packed into $K_5$ which is a subgraph of $K_6$. In the general case we take a minimal set of $C_3$'s and $C_6$'s to pack the remaining triangles. Pack these and then pack each of our minimal subset of cycles into the triangles and squares of the construction. It can be shown that we need to split at most one cycle and hence need only link an extra cycle to at most one of the squares. The result follows from part 3 of Lemma 3 with $l_1 = l = 4$ and $k \leq 2$.

3. The cases $s_T < r/2 - 2$, $s_T + p_T < r/2$ or $N = 16$.
Use the construction above with $s = s_T$, $p = p_T$, $t = r - 2s - 2p \geq 0$. Pack all cycles that use 4's or 5's, then continue with the $C_3$'s and $C_6$'s until we have packed $G_{2,r}.C_l$. In all these packings, $t \geq 2(s-1)$, so at most one extra cycle needs to be linked (at most two 3's occur in each cycle written with 4's and none occur in cycles written with 5's). The linked cycle is of length 3 or 6 (if it exists) and all unpacked cycles are $C_3$'s and $C_6$'s. These can be packed together with $C_l$ into $K_{N-2}$ by part 1 of Lemma 7 when $N > 8$. We pack the $C_3$'s and $C_6$'s into the sequence of triangles. Since the sequence is circular and meets every vertex, we may start at any vertex and the cycle linked to $G_{2,r}.C_l$ can be connected up properly. For $N = 8$ use parts 3 and 4 of Lemma 7 instead.

4. The cases $s_T + p_T \geq r/2$, $p_T \geq 3$, $6 \leq N \leq 12$.
Since $p_T \geq 3$, $N \geq 8$ and $r/2 \geq 3$. Use the construction with $t = 0$, $s = \min(s_T, r/2-3)$, $l = p = r/2 - s \geq 3$. Pick a minimal subset of cycles that use at least $s$ 4's and $p$ 5's between them. Pack these into $G_{2,r}.C_l$ as before. We may have to link extra cycles to the squares and pentagons in the construction. If $l > 3$ then all the 4's are used and the only extra cycles are 3's and 5's linked to 4's. There will be at most $s$ such cycles, but $s+l = r/2 \leq 5$, so if $l = 5$ then $s = 0$ and if $l = 4$, $s \leq 1$. Finally if $l = 3$ we may need to link cycles to all but one of the 4's and 5's, so we may need to link $s + p - 1 \leq 4$ cycles. (We will never need to link cycles to all the squares and pentagons since 5's can only be linked to 4's.) These can be linked to distinct vertices $b_j$ as in part 1. In all cases we are done by part 3 of Lemma 3.

We have now covered all possible cases, so the result follows. □

## 6. Packing Trails of Triangles in $G_{2n,r}$

In this section we shall pack some graphs of the form $G_{2n,r}$ with trails of triangles. If we have an exact packing of $G_{2n,r}$ with triangles, then each edge belongs to a unique triangle. Recall that in Section 2 we defined trails and cycles of triangles as trails and cycles (of edges) in which each edge belongs to a distinct triangle. These trails will be used in two ways. One is when we have a very large number of cycles of length divisible by three. These cycles can be packed into such a trail by a method analogous to Lemma 5. (A similar method was used in [1] to prove Theorem 1 when all the cycle lengths are divisible by 3). However, we shall also use these trails in Section 7 to construct packings of octahedra and related graphs into larger $G_{2n,r}$. We shall then pack these trails of octahedra with cycles of arbitrary lengths.

In Section 7 we will need to be careful about where the trail ends, and we may have to pack a cycle of triangles separately. Unfortunately, this introduces many technicalities into the proofs, which could be done much more simply otherwise.

Packings of $G_{2n,r}$ with triangles exist whenever $2n > r$ and $|E(G_{2n,r})| \equiv 0 \bmod 3$. We shall not prove this here in full generality, (for the cases $r \equiv 1, 3 \bmod 6$ see [8], for the general case see [10] and [7]), but restrict ourselves to some specific $r$ and $n$. Let $r = r(N)$

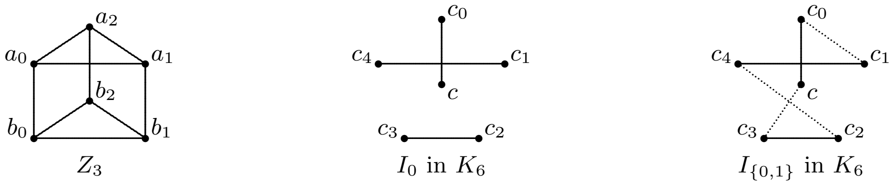



*Figure 6.* Examples of $Z_m$ and $I_S$.

be defined by

$$r(N) = \begin{cases} r(N+1)-1, & N \equiv 0 \bmod 2, \\ \frac{1}{2}(N-1), & N \equiv 3 \bmod 4, \\ \frac{1}{2}(N-7), & N \equiv 1 \bmod 4, N \geq 13, \\ 3, & N = 9, \\ \text{Undefined}, & N = 0, 1, 4, 5. \end{cases}$$

For these values of $r$, define $n$ by $N = 2n + r$. We shall pack triangles into $G_{2n,r}$.

Define a *prism* $Z_m$, $m \geq 3$, to be a 3-regular graph on $2m$ vertices $\{a_i, b_i : i \in \mathbb{Z}/m\mathbb{Z}\}$. The edges of $Z_m$ are those of the form $a_i b_i$, $a_i a_{i+1}$ and $b_i b_{i+1}$, so $Z_m$ is two cycles of length $m$ joined by a 1-factor. Define 1-factors of $K_{2m}$ by labeling the vertices as $\{c\} \cup \{c_i : i \in \mathbb{Z}/(2m-1)\mathbb{Z}\}$ and setting

$$I_s = \{c_i c_j : i + j \equiv s \bmod 2m-1\} \cup \{c c_i : 2i \equiv s \bmod 2m-1\}, \qquad s \in \mathbb{Z}/(2m-1)\mathbb{Z}.$$

For $S \subseteq \mathbb{Z}/(2m-1)\mathbb{Z}$, let $I_S$ be the edge disjoint union of $I_s$ for $s \in S$. In particular $I_{\{0,\dots,2m-2\}} = K_{2m}$.

**Lemma 9.**

1 *For all $s$, $I_{\{s,s+1\}}$ is a Hamiltonian cycle of $K_{2m}$.*

2 *Both $I_{\{0,1,2\}}$ and $Z_m$ are edge disjoint unions of three 1-factors, two of which form a Hamiltonian cycle.*

3 *$C_{2m} + E_2$ can be packed exactly with a cycle of triangles. It can also be packed exactly with a trail of triangles ending at a vertex $v$ of $E_2$ and starting with a triangle that meets the other vertex of $E_2$. Moreover, all but one of the vertices of $C_{2m}$ occur as interior vertices of the trail (of edges) corresponding to the trail of triangles.*

4 *Both $Z_m + E_3$ and $I_{\{0,1,2\}} + E_3$ can be packed exactly with a trail of triangles ending at a vertex $v$ of $E_3$ and starting with a triangle that meets one of the other two vertices of $E_3$. Moreover, all but at most one of the vertices of $Z_m$ or $I_{\{0,1,2\}}$ occur as interior vertices of the trail (of edges) corresponding to the trail of triangles.*

**Proof.**

1. By relabeling $c_i$ as $c_{i-s_0}$ where $2s_0 \equiv s \bmod 2m-1$ we can assume $s = 0$. Now

$$I_{\{0,1\}} = C(c_m, c_{m-1}, c_{m+1}, c_{m-2}, c_{m+2}, \dots, c_2, c_{2m-2}, c_1, c_0, c).$$

2. The cycle $C(a_1, a_2, \dots, a_{2m-1}, a_0, b_0, b_{2m-1}, \dots, b_2, b_1)$ is a Hamiltonian cycle of $Z_m$

and the remaining edges $I = \{a_i b_i : 2 \leq i < m\} \cup \{a_0 a_1, b_0 b_1\}$ form a 1-factor. Since the cycle is of even length, it can be decomposed into two 1-factors. For $I_{\{0,1,2\}}$ the subgraph $I_{\{0,1\}}$ forms a Hamiltonian cycle and $I_2$ is the remaining 1-factor.

3. Let the cycle be $C(a_1, \ldots, a_{2m})$ and let the two other vertices be $u$ and $v$. The sequence of triangles $C(u, a_1, a_2), C(v, a_2, a_3), C(u, a_3, a_4), \ldots, C(v, a_{2m}, a_1)$ gives a cycle of triangles corresponding to $C(a_1, \ldots, a_{2m})$ and a trail of triangles corresponding to $P(u, a_2, \ldots, a_{2m}, v)$. All the vertices of $C_{2m}$ except $a_1$ occur in the interior of the trail.

4. Let the three vertices of $E_3$ be $\{u, v, w\}$. Relabel the vertices of the Hamiltonian cycle of $Z_m$ or $I_{\{0,1,2\}}$ given in part 2 as $C(u_1, v_1, u_2, v_2, \ldots)$. We obtain triangles $C(u, u_i, v_i)$, $C(v, v_i, u_{i+1})$ and triangles formed by joining the remaining 1-factor $I$ to $w$. The subgraph $G = \{uu_i\} \cup \{vv_i\} \cup I$ contains one edge from each triangle and has degree two at each vertex except possibly at $u$ and $v$. We now show that it is also connected (except for the isolated vertex $w$). In the $Z_m$ case, $I$ joins $a_2 = v_1$ and $b_2 = u_m$. In the $I_{\{0,1,2\}}$ case, $I_2$ joins $c_0 = u_m$ and $c_2 = v_{m-2}$. In both cases, all the $u_i$ are connected to $u$ and all the $v_i$ are connected to $v$, so $G$ is connected. Therefore $G$ has an Eulerian trail. If $m$ is odd, the trail starts at $u$ and ends at $v$. If $m$ is even, we get an Eulerian circuit. For even $m$, remove one edge $xy$ of $I$, and replace it with an edge $xw$. This gives a graph $G'$ with an Eulerian trail ending at $w$ and starting with an edge from a triangle meeting either $u$ or $v$. In both cases, each edge of the trail lies in a unique triangle, so the result follows. Note that every vertex of $Z_m$ or $I_{\{0,1,2\}}$ except possibly $y$ has degree two in $G$ or $G'$, so occurs as an interior vertex of the trail. □

**Theorem 10.** *If $N = 2n+r$ and $r = r(N)$ then for $N \geq 6$ there exists an exact packing of triangles into $G_{2n,r}$ which forms a trail of triangles with the last vertex $v \in R$ and the first triangle meeting $R \setminus \{v\}$. If $N \geq 9$, then there is also an exact packing which is an edge-disjoint union of a Hamiltonian cycle of triangles in $M$ and such a trail.*

By a Hamiltonian cycle of triangles in $M$ we mean that we can replace each triangle with a single edge so as to get a cycle of length $2n$ inside the subset $M$ of vertices. We do *not* require each triangle to have all its vertices in $M$.

**Proof.**

1. The cases $N \equiv 2, 3 \bmod 4$, $N \geq 6$.

In these cases $r = 2n - 1$ if $N$ is odd and $r = 2n - 2$ if $N$ is even. Label the vertices of $R$ as $v_0, \ldots, v_{r-1}$ and write $G_{2n,r}$ as an edge-disjoint union of $I_s + E_{\{v_s\}}$, where the $I_s$ are the 1-factors of $K_M$ constructed above (with $m = n$). If $N$ is even, $I_{2n-2}$ will not be used, so this will be the missing 1-factor $I_M$ of $K'_M$ in $G_{2n,r} = K'_M + E_R$. For even $N$, use part 3 of Lemma 9 to pack $I_{\{0,1\}} + E_{\{v_0,v_1\}}$ with a trail of triangles. For odd $N$, use part 4 of Lemma 9 to pack $I_{\{0,1,2\}} + E_{\{v_0,v_1,v_2\}}$ with a trail of triangles. In both cases, pair up the remaining subgraphs as $I_{\{s,s+1\}} + E_{\{v_s,v_{s+1}\}}$ and use part 3 of Lemma 9 to pack them with Hamiltonian cycles of triangles in $M$. By inserting cycles before the last triangle of the trail we can always combine any surplus Hamiltonian cycles of triangles with the trail to get a longer trail starting and ending with the same triangles. For $N \geq 9$ (and $N \neq 13$), $r > 2$ and we have at least one such Hamiltonian cycle.

2. The cases $N \equiv 0, 1 \bmod 4$, $N \geq 13$.
Split $M = M_a \cup M_b$ into two subsets of $n$ vertices and label the vertices as $M_a = \{a_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ and $M_b = \{b_i : i \in \mathbb{Z}/n\mathbb{Z}\}$. For $s \in \mathbb{Z}/n\mathbb{Z}$ let $I'_s$ be the 1-factor $\{a_i b_{i+s} : i \in \mathbb{Z}/n\mathbb{Z}\}$. If $n$ is odd, define 2-factors $S_s = \{a_i a_{i+s}, b_i b_{i+s} : i \in \mathbb{Z}/n\mathbb{Z}\}$ for $1 \leq s < n/2$. If $n$ is even, we can decompose $K_n = K_{2m}$ into a 1-factor and $\frac{1}{2}(n-2)$ Hamiltonian cycles as above. Combining 1-factors for $M_a$ and $M_b$ gives a 1-factor $I_*$ of $M$. Combining corresponding Hamiltonian cycles gives 2-factors $S_s$, $1 \leq s < n/2$ of $M$ (each a union of two $n$-cycles). In both $n$ even and $n$ odd cases we may assume that $S_1$ consists of cycles $C(a_0, a_1, \ldots, a_{n-1})$ and $C(b_0, b_1, \ldots, b_{n-1})$. The edge disjoint union of $S_1$, $I'_1$, $I'_{n-1}$, $I'_2$ and $I'_{n-2}$ can be packed with the triangles

$$C(a_i, a_{i+1}, b_{i+2}), \quad C(b_i, b_{i+1}, a_{i+2}), \qquad i \in \mathbb{Z}/n\mathbb{Z}.$$

These triangles form a Hamiltonian cycle of triangles in $M$ corresponding to the cycle $C(a_0, b_2, a_1, b_3, \ldots)$.

If $n$ is odd, then the edge union of any $S_s$ with any $I'_t$ is a vertex disjoint union of $n/m$ prisms $Z_m$ where $n/m = \gcd(s, n)$. It is therefore a union of three 1-factors. There are $\frac{1}{2}(n-3)$ remaining $S_s$'s and $n-4$ remaining $I'_t$'s. Since $n \geq 5$ when $N \geq 13$, $n - 4 \geq \frac{1}{2}(n-3)$ and so all remaining edges can be decomposed into 1-factors. These then form $2n-7$ 1-factors, and three of these 1-factors can be chosen to form a prism $Z_n$ made up of $S_s$ and $I_0$ for some $s$ with $\gcd(s, n) = 1$ (e.g., $s = 2$).

If $n$ is even then $S_s$ is a union of two 1-factors. These and the remaining $I'_t$ and $I_*$ give $2n-7$ 1-factors. Three of these 1-factors can be chosen to form a prism $Z_n$ made up from $S_s$ and $I_0$ for some $s$ as before.

If $N$ is odd, part 4 of Lemma 9 gives a trail of triangles in $Z_n + E_3$ where $E_3$ is any three vertices in $R$. If $N$ is even, write $Z_n$ as a union of a Hamiltonian cycle and a 1-factor. Remove this 1-factor, it will be the missing 1-factor $I_M$ of $K'_M$. The remaining Hamiltonian cycle when joined to two vertices in $R$ gives a trail of triangles by part 3 of Lemma 9.

In both cases, the remaining edges of $G_{2n,r}$ can be decomposed into cycles of triangles by pairing the remaining 1-factors of $M$, joining each pair to a pair of points in $R$ and using part 3 of Lemma 9. The the cycles (of edges) corresponding to these cycles of triangles each meet an interior vertex of the trail corresponding to the trail of triangles (since the all but at most one of the vertices of $M$ are such a vertex). Hence we can insert the cycles into the trail giving a longer trail with the same initial and final triangles. Doing this with each excess cycle in turn gives the result.

3. The cases $N = 8$ and $N = 9$.
Label the vertices $M = \{a_0, a_1, a_2, b_0, b_1, b_2\}$ and $R = \{v_0, v_1\}$ ($N = 8$) or $R = \{v_0, v_1, v_2\}$ ($N = 9$). We can pack $G_{6,3}$ with triangles $T_a = C(a_0, a_1, a_2)$, $T_b = C(b_0, b_1, b_2)$ and $T_{ijk} = C(a_i, b_j, v_k)$ for all $j \equiv i + k \bmod 3$. If we remove all $T_{ij2}$'s we get a packing of $G_{6,2}$. For $N = 8$ we have the trail of triangles $T_{000}T_{110}T_{011}T_{121}T_aT_{220}T_bT_{201}$, corresponding to $P(a_0, v_0, b_1, v_1, a_1, a_2, b_2, b_0, v_1)$. For $N = 9$, we have a cycle $T_aT_{000}T_{102}T_{121}T_bT_{212}$ and a trail $T_{110}T_{011}T_{201}T_{220}T_{022}$, corresponding to the cycle $C(a_2, a_0, b_0, a_1, b_2, b_1)$ and the trail $P(v_0, b_1, v_1, a_2, b_2, v_2)$ respectively. □

**Corollary 11.** *If $N \geq 6$, $r = r(N)$, $N = 2n + r$ and $T_S = \frac{1}{3}|E(G_{2n,r})|$, then for any $s$ with $0 < s \leq T_S$ it is possible to pack $G_{2n,r}$ exactly with triangles so that $s$ of them form a trail of triangles $C(v_i, u_i, v_{i+1})$, $1 \leq i \leq s$ corresponding to a trail $P(v_1, \ldots, v_{s+1})$ with $v_{s+1} \in R$. Also*

*(a) If $N = 7$ or $N \geq 9$ and $s \geq 2n$ then we can make $v_1, \ldots v_{2n}$ a permutation of vertices of $M$.*

*(b) If $s < T_S$ then one of the remaining triangles can be made to meet $R \setminus \{v_{s+1}\}$.*

*(c) If $N \geq 9$ and $2n \leq s < T_S$ then both (a) and (b) can be made to hold simultaneously.*

**Proof.** Use the first part of Theorem 10 to get a trail of triangles and take the last $s$ triangles in this trail. This gives the first part and (b) since if $s < T_S$ then the first triangle of the original trail is not used and meets $R \setminus \{v_{s+1}\}$. For $N \geq 9$ use Theorem 10 to obtain a Hamilton cycle and a trail. Use the last $s - 2n$ triangles of the trail (or just the last one if $s = 2n$). The first of these triangles must meet $M$ in at least two vertices, so we can attach the Hamiltonian cycle (or a Hamiltonian path if $s = 2n$) onto the front of this trail. Part (a) and (c) now follow provided $N \neq 7$. Finally, for $N = 7$ we have the following packing of $G_{4,3}$ with triangles where $M = \{a_0, a_1, a_2, a_3\}$ and $R = \{v_0, v_1, v_2\}$:

$$C(a_0, a_1, v_0), C(a_2, a_3, v_0), C(a_0, a_2, v_1), C(a_1, a_3, v_1), C(a_0, a_3, v_2), C(a_1, a_2, v_2).$$

The following trails give suitable trails of triangles in (a) for $N = 7$:

$P(a_0, a_1, a_2, a_3, v_1, a_0, v_2)$ for $s = T_S = 6$,
$P(a_0, a_1, a_2, a_3, a_0, v_1)$ for $s = 5$ and
$P(a_0, a_1, a_2, a_3, v_1)$ for $s = 4$.

□

## 7. Packing Trails of Octahedra into $G_{2n,r}$.

We shall now pack some graphs of the form $G_{2n,r}$ with trails of octahedra and related graphs. First, however, we need to define how these graphs are linked up.

Recall that for some graphs we can define initial and final links as (ordered) sets of vertices, and $G_1.G_2$ will identify the final link of $G_1$ with the initial link of $G_2$. The initial link of the resultant graph is that of $G_1$ and the final link is that of $G_2$. Similarly, the initial link of $G_1 \cup G_2$ is that of $G_1$ and the final link is that of $G_2$. We also write $G^{\cdot n}$ for $G.G \ldots G$ and $G^{\cup n}$ for $G \cup \ldots \cup G$ where there are $n$ copies of $G$.

We have also defined $O$ to be the graph of an octahedron, so $O = K_6' = E_2 + E_2 + E_2$. The first $E_2$ will be the initial link and last $E_2$ will be the final link of $O$. By symmetry, it does not matter which $E_2$'s are chosen, or the order of the vertices in either link. Define $K_5 = K_4 + E_1$ to have initial and final link both equal to the same single vertex $E_1$. Define $W = G_{4,4} = C_4 + E_4$. The $E_4$ will be both the initial and final link (with the vertices in the same order). Define the triangle $T = K_2 + E_1$ with $K_2$ both the initial and final link (with vertices in the same order). Note that this differs from the links of a triangle as described in Section 3.

**Theorem 12.**

*1* *If* $N \equiv 2 \bmod 4$ *and* $N \geq 14$ *then there are exact packings of* $O^{\cup a} \cup O^{\cdot b}$ *into* $G_{4n,2r}$ *with* $4n + 2r = N$, $r = r(\frac{1}{2}N) \geq 3$ *and the final link mapped into* $R$.

*2* *If* $N \equiv 3 \bmod 4$ *and* $N \geq 15$ *then there are exact packings of* $O^{\cup a} \cup (T.O)^{\cdot 2n}.O^{\cdot b}$ *into* $G_{4n,2r+1}$ *with* $4n+2r+1 = N$, $r = r(\frac{1}{2}(N-1)) \geq 3$ *and the final link mapped into* $R$. *In addition, the packing maps the non-link vertices of each* $T$ *to a single vertex in* $R$.

*3* *If* $N \equiv 1 \bmod 4$ *and* $N \geq 13$ *then there are exact packings of* $K_5^{\cdot n} \cup O^{\cup a} \cup O^{\cdot b}$ *into* $G_{4n,2r+1}$ *with* $4n+2r+1 = N$, $r = r(\frac{1}{2}(N-1)) \geq 2$ *and the final link mapped into* $R$. *Also, the link vertex of* $K_5^{\cdot n}$ *is packed as a vertex of* $R$ *distinct from the image of the vertices of any* $O$.

*4* *If* $N \equiv 0 \bmod 4$ *and* $N \geq 16$ *then there are exact packings of* $W^{\cdot n} \cup O^{\cup a} \cup O^{\cdot b}$ *into* $G_{4n,2r+4}$ *with* $4n+2r+4 = N$, $r = r(\frac{1}{2}(N-4)) \geq 2$ *and the final link mapped into* $R$. *Also, the link vertices of* $W^{\cdot n}$ *are packed as vertices of* $R$ *disjoint from the image of the vertices of any* $O$.

*In each case such packings exist for all* $a$, $b$ *with* $a + b = \frac{1}{3}|E(G_{2n,r})|$, $a, b \geq 0$. *Also, if* $a > 0$, $N \neq 15$, *then at least one* $O$ *in the* $O^{\cup a}$ *can be required to have its final link in* $R$ *and disjoint from the final link of* $O^{\cdot b}$ *(or of* $(T.O)^{\cdot 2n}$ *in part 2 if* $b = 0$*).*

**Proof.**

1. Set $r = r(\frac{1}{2}N)$, $2n + r = \frac{1}{2}N$. By Corollary 11, we can pack $G_{2n,r}$ with a trail of triangles with the final vertex in $R$. Replace each vertex $v$ of $G_{2n,r}$ by a pair of vertices $v_0$, $v_1$, and each edge $uv$ by four edges $u_i v_j$. The resulting graph is just $G_{4n,2r}$. The triangles become octahedra and a trail of triangles becomes a packing of linked octahedra $O^{\cdot m}$. The result now follows from Corollary 11.

2. As in 1, construct $G_{4n,2r}$ with $2n+r = \frac{1}{2}(N-1)$. Add one vertex $v$ to $R$ and join it to the missing 1-factor $I_M$ of $M$ in $G_{4n,2r}$. This gives $G_{4n,2r+1}$. The extra triangles can be linked to the first $2n$ links in the $O$-trail by part (a) of Corollary 11 since $\frac{1}{2}(N-1) = 7$ or $\frac{1}{2}(N-1) \geq 9$. Provided $N \neq 15$ (so $\frac{1}{2}(N-1) \geq 9$) we can also ensure that one of the isolated $O$'s has a link that maps into $R$ as a set disjoint from the final link of the trail.

3. As in 1, construct $G_{2n,r}$ and double up vertices. This time $r$ is even, so to obtain $G_{4n,2r+1}$ we must join the extra vertex $v$ to $n$ vertex disjoint copies of $K_4$ in $K_M = K_{4n}$. (Each of these $K_4$'s comes from doubling vertices in a component of $I_M$.) This gives $K_5^{\cdot n}$. The rest of the proof is similar to part 1.

4. As in 3, except that we add the four extra vertices to $R$ and join them to $n$ vertex disjoint copies of $K'_4 = C_4$ in $K_M = K_{4n}$. This gives $n$ copies of $C_4 + E_4$ with the $E_4$'s linked. In other words $W^{\cdot n}$. □

We now have the decompositions of $G_{n,r}$ into octahedra and other small graphs as described in Section 3.

## 8. Packing Cycles into $O$ and $T.O$

Having packed trails of octahedra into suitable $G_{n,r}$, we now need to pack cycles into these octahedra. In general it will not be possible to group the cycles into combinations

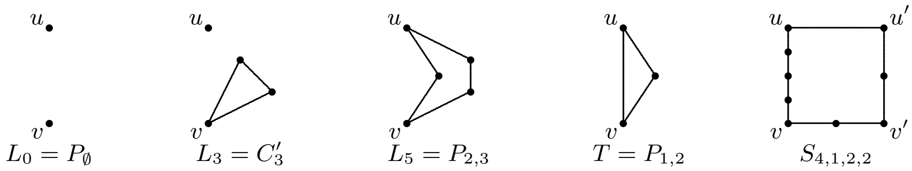



*Figure 7.* Graphs $L_n$, $T$ and $S_{a,b,c,d}$.

of length $|E(O)| = 12$. Therefore we shall attach cycles to the link vertices of $O$ to allow overlaps from one octahedron to the next. These overlaps are the $L_n$ defined below.

For a path $P_n$ of length $n$ with endpoints $u$ and $v$, we shall make $(u, v)$ both the initial and final link of $P_n$. Write $C_n' = C_n \cup E_1$ to denote a cycle of length $n$ together with an extra independent vertex. The pair $(u, v)$ will be both the initial and final link of $C_n'$ where $u$ is the independent vertex in $E_1$ and $v$ is any vertex of the cycle $C_n$. The graph $P_{a_1,\dots,a_r} = P_{a_1}.P_{a_2}\dots P_{a_r}$ will be a graph with specified link vertices $(u, v)$ consisting of $r$ internally vertex disjoint paths of lengths $a_1, \dots, a_r$ from $u$ to $v$. In the special case when $r = 0$ we write $P_\emptyset$ for the empty graph $E_2$ on $\{u, v\}$. We write $S_{a,b,c,d}$ for a cycle with initial link $(u, v)$, final link $(u', v')$ and four internally vertex disjoint paths connecting these four vertices as follows. A path of length $a$ connects $u$ and $v$, a path of length $b$ connects $u$ and $u'$, a path of length $c$ connects $v$ and $v'$ and a path of length $d$ connects $u'$ and $v'$ (see Figure 7)

**Definition.** The graphs $L_n$ are defined as

$$L_0 = P_\emptyset, \quad L_3 = C_3', \quad L_4 = P_{2,2}, \quad L_5 = P_{2,3}, \quad \text{and} \quad L_n = P_{4,n-4} \text{ for } n \geq 7.$$

The graph $L_6$ will be defined as *either* $P_{3,3}$ *or* $P_{4,2}$. By this we mean that whenever we pack a graph involving $L_6$ into another graph, we require that packings exists for both choices of $L_6$. On the other hand, if we pack a graph into $L_6$ then we only require a packing exists for some choice of $L_6$. We define the triangle $T = P_{1,2}$ as before. Note that $L_n$ is a cycle for $n \geq 3$, and we can always pack $C_n' \mapsto L_n$ with initial and final links matching.

**Lemma 13.** *The following can be packed into $O$ with initial and final links matching:*

$$P_{2,2,2,2} \cup P_{2,2}, \quad P_\emptyset \cup P_{3,3,3,3}, \quad S_{4,1,1,3}.C_3', \quad S_{4,1,2,2}.C_3',$$
$$L_n.C_3' \cup L_{9-n},\ (4 \leq n \leq 6) \quad \textit{and} \quad L_n \cup L_{12-n},\ (3 \leq n \leq 9).$$

**Proof.** These packings are trivial to find, but their descriptions are somewhat tedious. Number the vertices of $O$ from 0 to 5 so that $O = E_{\{0,1\}} + E_{\{2,3\}} + E_{\{4,5\}}$ with $(0, 1)$ the initial link and $(4, 5)$ the final link. We pack the paths and cycles as follows:

$$P_{2,2,2,2} \cup P_{2,2} \mapsto \{P(0, 2, 1), P(0, 3, 1), P(0, 4, 1), P(0, 5, 1); P(4, 2, 5), P(4, 3, 5)\}$$
$$P_\emptyset \cup P_{3,3,3,3} \mapsto \{; P(4, 0, 2, 5), P(4, 2, 1, 5), P(4, 1, 3, 5), P(4, 3, 0, 5)\}$$

$$\begin{aligned}
S_{4,1,1,3}.C_3' &\mapsto \{P(0,2,4,3,1),P(0,4),P(1,5),P(4,1,2,5);P(5,0,3,5)\}\\
S_{4,1,2,2}.C_3' &\mapsto \{P(0,2,4,3,1),P(0,4),P(1,2,5),P(4,1,5);P(5,0,3,5)\}\\
P_{4,2}.C_3'\cup C_3' &\mapsto \{P(0,2,4,3,1),P(0,4,1);P(1,2,5,1);P(5,0,3,5)\}\\
P_{3,3}.C_3'\cup C_3' &\mapsto \{P(0,4,3,1),P(0,2,4,1);P(1,2,5,1);P(5,0,3,5)\}\\
P_{2,3}.C_3'\cup P_{2,2} &\mapsto \{P(0,3,1),P(0,2,4,1);P(1,2,5,1);P(4,3,5),P(4,0,5)\}\\
P_{2,2}.C_3'\cup P_{2,3} &\mapsto \{P(0,3,1),P(0,4,1);P(1,2,5,1);P(4,3,5),P(4,2,0,5)\}\\
P_{4,5}\cup C_3' &\mapsto \{P(0,2,4,3,1),P(0,4,1,2,5,1);P(5,0,3,5)\}\\
P_{4,4}\cup P_{2,2} &\mapsto \{P(0,2,4,3,1),P(0,3,5,2,1);P(4,1,5),P(4,0,5)\}\\
P_{4,3}\cup P_{2,3} &\mapsto \{P(0,2,4,3,1),P(0,5,2,1);P(4,1,5),P(4,0,3,5)\}\\
P_{4,2}\cup P_{4,2} &\mapsto \{P(0,4,2,5,1),P(0,2,1);P(4,1,3,0,5),P(4,3,5)\}\\
P_{3,3}\cup P_{4,2} &\mapsto \{P(0,2,4,1),P(0,3,5,1);P(4,3,1,2,5),P(4,0,5)\}\\
P_{3,3}\cup P_{3,3} &\mapsto \{P(0,4,3,1),P(0,3,5,1);P(4,1,2,5),P(4,2,0,5)\}
\end{aligned}$$

In each case the edge union of the trails on the right is $O$. In most cases, the decomposition is a minor variant of a preceding one, so can be checked easily. Note that by symmetry, a packing of $L_3\cup L_9$, for example, follows from a packing of $L_9\cup L_3$. Also note that whenever $L_6$ is used, both versions have been checked. □

**Theorem 14.** *Suppose that either $m+\sum m_i\geq 15$ or $m+\sum m_i=12$ with $m\geq 0$, $m\neq 1,2$, $m_i\geq 5$, $m_i\neq 6$. For some subset $S$ and some $m'$ we can pack $L_m\cup(\cup_{i\in S}C_{m_i})$ into $O.L_{m'}$ exactly with initial link matching, except in the cases when $m\in\{0,4,5,9\}$ and all the $m_i=5$. If $m=8$ we can also pack $P_{2,2,2,2}\cup(\cup_{i\in S}C_{m_i})$ in the same way.*

**Proof.** See Table 1 for the packings used. In each case we can pack graph (A) into (B) by linking up suitable paths, with the initial link of $L_m$ in (A) mapped to the initial link in (B). (For ease of checking, the underlined cycles in (A) are packed into the underlined paths and cycles in (B).) We then pack (B) into (C) by Lemma 13. In the case †, the $P_3$ part of $L_7=P_{4,3}$ meets vertex 5 of $O$ (see proof of Lemma 13) so can be linked with the $C_3'$ to pack $L_{10}=P_{4,6}$. It is easy to check that if $m>0$ and we are not in one of the exceptional cases, then we must have a subset of one of the forms in Table 1. If $m=0$, pack some $C_{m_{i_0}}$ into $L_{m_{i_0}}$ first and then use the result with $m>0$. If there are $C_5$'s, make sure to use one as our $C_{m_{i_0}}$. The only additional case that cannot be packed is when all the $m_i=5$, which is included in the list of exceptional cases. □

We now repeat the previous two results with $O$ replaced by $T.O$.

**Lemma 15.** *The following can be packed into $T.O$ with initial and final links matching:*

$$L_8\cup P_{5,2},\quad L_8\cup P_{4,3},\quad L_7\cup P_{6,2},\quad L_7\cup P_{5,3},\quad P_{3,3}\cup P_{7,2},\quad P_{3,3}\cup P_{6,3},$$
$$P_{3,3}\cup P_{5,4},\quad L_5\cup P_{4,3}.C_3',\quad L_5\cup P_{5,2}.C_3',\quad L_4\cup P_{4,3,2,2},\quad L_4\cup P_{5,2,2,2}.$$

*Table 1.* Packings into $O.L_n$ used in Theorem 14.

| $(A)$ | $(B)$ | $(C)$ | Conditions |
|---|---|---|---|
| $L_m$ | $S_{4,1,2,2}.C'_3.C'_{m-12}$ | $O.L_{m-12}$ | $m \geq 15$ |
| $L_{14} \cup C_{\underline{n}}$ | $S_{4,1,2,\underline{2}}.C'_3.P_{4,\underline{n-2}}$ | $O.L_{n+2}$ | $n \geq 5$ |
| $L_{13} \cup C_{\underline{n}}$ | $S_{4,1,1,\underline{3}}.C'_3.P_{4,\underline{n-3}}$ | $O.L_{n+1}$ | $n \geq 5$ |
| $L_{12}$ | $S_{4,1,2,2}.C'_3$ | $O.L_0$ | |
| $L_{11} \cup C_{\underline{n}}$ | $S_{4,1,2,\underline{2}}.C'_{\underline{3}}.P_{4,\underline{n-5}}$ | $O.L_{n-1}$ | $n \geq 7$ |
| $L_{11} \cup C_{\underline{5}}$ | $S_{4,1,1,\underline{3}}.C'_3.P_{2,\underline{2}}$ | $O.L_4$ | |
| $L_{10} \cup C_{\underline{n}}$ | $S_{4,1,1,\underline{3}}.C'_{\underline{3}}.P_{4,\underline{n-6}}$ | $O.L_{n-2}$ | $n \geq 8$ |
| $L_{10} \cup C_{\underline{7}}$ | $S_{4,1,2,\underline{2}}.C'_{\underline{3}}.P_{\underline{2},3}$ | $O.L_5$ | |
| $L_{10} \cup C_{\underline{5}}$ | $L_7 \cup L_{\underline{5}}.C'_3$ | $O.L_3$ | See text† |
| $L_m \cup C_{\underline{n}}$ | $L_m \cup L_{\underline{12-m}}.C'_{\underline{n+m-12}}$ | $O.L_{n+m-12}$ | $3 \leq m \leq 9, n+m \geq 15$ |
| $L_m \cup C_{\underline{n}}$ | $L_m \cup L_{\underline{n}}$ | $O.L_0$ | $3 \leq m \leq 9, n+m = 12$ |
| $L_8 \cup C_{\underline{5}} \cup C_5$ | $L_8 \cup P_{\underline{2},2}.P_{\underline{3},3}$ | $O.L_6$ | |
| $P_{2,2,2,2} \cup C_{\underline{n}}$ | $P_{2,2,2,2} \cup P_{\underline{2},\underline{2}}.C'_{\underline{n-4}}$ | $O.L_{n-4}$ | $n \geq 7$ |
| $P_{2,2,2,2} \cup C_{\underline{5}} \cup C_5$ | $P_{2,2,2,2} \cup P_{\underline{2},2}.P_{\underline{3},3}$ | $O.L_6$ | |
| $L_7 \cup C_{\underline{7}} \cup C_n$ | $L_7 \cup P_{2,\underline{3}}.P_{\underline{4},n-2}$ | $O.L_{n+2}$ | $n \geq 5$ |
| $L_6 \cup C_{\underline{8}} \cup C_n$ | $L_6 \cup P_{\underline{4},2}.P_{\underline{4},n-2}$ | $O.L_{n+2}$ | $n \geq 5$ |
| $L_6 \cup C_{\underline{7}} \cup C_n$ | $L_6 \cup P_{\underline{3},3}.P_{\underline{4},n-3}$ | $O.L_{n+1}$ | $n \geq 5$ |
| $L_6 \cup C_{\underline{5}} \cup C_5$ | $L_6 \cup P_{\underline{3},3}.P_{\underline{2},2}$ | $O.L_4$ | |
| $L_5 \cup C_{\underline{9}} \cup C_n$ | $L_5 \cup C'_{\underline{3}}.P_{\underline{2},2}.P_{\underline{4},n-2}$ | $O.L_{n+2}$ | $n \geq 5$ |
| $L_5 \cup C_{\underline{8}} \cup C_n$ | $L_5 \cup P_{\underline{4},3}.P_{\underline{4},n-3}$ | $O.L_{n+1}$ | $n \geq 5$ |
| $L_4 \cup C_{\underline{10}} \cup C_n$ | $L_4 \cup P_{\underline{2},\underline{2},\underline{2},2}.P_{\underline{4},n-2}$ | $O.L_{n+2}$ | $n \geq 5$ |
| $L_4 \cup C_{\underline{9}} \cup C_n$ | $L_4 \cup C'_{\underline{3}}.P_{\underline{2},3}.P_{\underline{4},n-3}$ | $O.L_{n+1}$ | $n \geq 5$ |
| $L_4 \cup C_{\underline{7}} \cup C_7$ | $L_4 \cup P_{\underline{4},4}.P_{\underline{3},3}$ | $O.L_6$ | |
| $L_4 \cup C_{\underline{5}} \cup C_n$ | $L_4 \cup C'_3.L_{\underline{5}}.C'_{n-3}$ | $O.L_{n-3}$ | $n \geq 7$ |
| $L_3 \cup C_{\underline{11}} \cup C_n$ | $L_3 \cup C'_{\underline{3}}.P_{\underline{4},2}.P_{\underline{4},n-2}$ | $O.L_{n+2}$ | $n \geq 5$ |
| $L_3 \cup C_{\underline{10}} \cup C_n$ | $L_3 \cup C'_{\underline{3}}.P_{\underline{3},3}.P_{\underline{4},n-3}$ | $O.L_{n+1}$ | $n \geq 5$ |
| $L_3 \cup C_{\underline{8}} \cup C_n$ | $L_3 \cup C'_3.P_{\underline{4},2}.P_{\underline{4},n-5}$ | $O.L_{n-1}$ | $n \geq 7$ |
| $L_3 \cup C_{\underline{7}} \cup C_n$ | $L_3 \cup C'_3.P_{\underline{3},3}.P_{\underline{4},n-6}$ | $O.L_{n-2}$ | $n \geq 8$ |
| $L_3 \cup C_{\underline{7}} \cup C_7$ | $L_3 \cup C'_3.P_{\underline{4},2}.P_{2,\underline{3}}$ | $O.L_5$ | |
| $L_3 \cup C_{\underline{5}} \cup C_n$ | $L_{\underline{5}}.C'_3 \cup L_4.C'_{n-4}$ | $O.L_{n-4}$ | $n \geq 7$ |
| $L_3 \cup C_{\underline{5}} \cup C_5 \cup C_{\underline{5}}$ | $L_{\underline{5}}.C'_3 \cup P_{2,\underline{2}}.P_{3,\underline{3}}$ | $O.L_6$ | |

**Proof.** Number the vertices of $O$ 0 to 5 as before and let $T = C(0,1,v)$ with initial and final link equal to $(0,1)$. We pack the paths as follows:

$$P_{4,4} \cup P_{5,2} \mapsto \{P(0,2,4,3,1), P(0,3,5,2,1); P(4,0,1,v,0,5), P(4,1,5)\}$$

$$P_{4,4} \cup P_{4,3} \mapsto \{P(0,2,4,3,1), P(0,3,5,2,1); P(4,1,v,0,5), P(4,0,1,5)\}$$

$$P_{4,3} \cup P_{6,2} \mapsto \{P(0,2,4,3,1), P(0,5,2,1); P(4,0,1,v,0,3,5), P(4,1,5)\}$$

$$P_{4,3} \cup P_{5,3} \mapsto \{P(0,2,4,3,1), P(0,5,2,1); P(4,1,v,0,3,5), P(4,0,1,5)\}$$

$$P_{3,3} \cup P_{7,2} \mapsto \{P(0,4,3,1), P(0,5,2,1); P(4,2,0,1,v,0,3,5), P(4,1,5)\}$$

$$P_{3,3} \cup P_{6,3} \mapsto \{P(0,2,4,1), P(0,5,2,1); P(4,0,1,v,0,3,5), P(4,3,1,5)\}$$

$$P_{3,3} \cup P_{5,4} \mapsto \{P(0,4,3,1), P(0,5,2,1); P(4,1,v,0,3,5), P(4,2,0,1,5)\}$$

$$P_{2,3} \cup P_{4,3}.C'_3 \mapsto \{P(0,v,1), P(0,4,2,1); P(4,1,0,2,5), P(4,3,1,5); P(5,0,3,5)\}$$

$$P_{2,3} \cup P_{5,2}.C'_3 \mapsto \{P(0,v,1), P(0,4,2,1); P(4,3,1,0,2,5), P(4,1,5); P(5,0,3,5)\}$$

*Table 2.* Packings into $T.O.L_n$ used in Theorem 16.

| $(A)$ | $(B)$ | $(C)$ | Conditions |
|---|---|---|---|
| $L_8 \cup C_{\underline{n}}$ | $L_8 \cup P_{\underline{5},\underline{2}}.C'_{\underline{n-7}}$ | $T.O.L_{n-7}$ | $n \geq 10$ |
| $L_8 \cup C_{\underline{9}} \cup C_n$ | $L_8 \cup P_{\underline{5},2}.P_{\underline{4},n-2}$ | $T.O.L_{n+2}$ | $n \geq 7$ |
| $L_8 \cup C_{\underline{8}} \cup C_n$ | $L_8 \cup P_{\underline{4},3}.P_{\underline{4},n-3}$ | $T.O.L_{n+1}$ | $n \geq 7$ |
| $L_8 \cup C_{\underline{7}}$ | $L_8 \cup P_{\underline{4},\underline{3}}$ | $T.O.L_0$ | |
| $L_7 \cup C_{\underline{n}}$ | $L_7 \cup P_{\underline{6},\underline{2}}.C'_{\underline{n-8}}$ | $T.O.L_{n-8}$ | $n \geq 11$ |
| $L_7 \cup C_{\underline{10}} \cup C_n$ | $L_7 \cup P_{\underline{6},2}.P_{\underline{4},n-2}$ | $T.O.L_{n+2}$ | $n \geq 7$ |
| $L_7 \cup C_{\underline{9}} \cup C_n$ | $L_7 \cup P_{\underline{5},3}.P_{\underline{4},n-3}$ | $T.O.L_{n+1}$ | $n \geq 7$ |
| $L_7 \cup C_{\underline{8}}$ | $L_7 \cup P_{\underline{5},\underline{3}}$ | $T.O.L_0$ | |
| $L_7 \cup C_{\underline{7}} \cup C_7$ | $L_7 \cup P_{\underline{5},3}.P_{4,\underline{2}}$ | $T.O.L_6$ | |
| $L_6 \cup C_{\underline{n}}$ | $P_{3,3} \cup P_{\underline{7},\underline{2}}.C'_{\underline{n-9}}$ | $T.O.L_{n-9}$ | $n \geq 12$ |
| $L_6 \cup C_{\underline{11}} \cup C_n$ | $P_{3,3} \cup P_{\underline{7},2}.P_{\underline{4},n-2}$ | $T.O.L_{n+2}$ | $n \geq 7$ |
| $L_6 \cup C_{\underline{10}} \cup C_n$ | $P_{3,3} \cup P_{\underline{6},3}.P_{\underline{4},n-3}$ | $T.O.L_{n+1}$ | $n \geq 7$ |
| $L_6 \cup C_{\underline{9}}$ | $P_{3,3} \cup P_{\underline{6},\underline{3}}$ | $T.O.L_0$ | |
| $L_6 \cup C_{\underline{8}} \cup C_n$ | $P_{3,3} \cup P_{5,\underline{4}}.P_{\underline{4},n-5}$ | $T.O.L_{n-1}$ | $n \geq 7$ |
| $L_6 \cup C_{\underline{7}} \cup C_7$ | $P_{3,3} \cup P_{5,\underline{4}}.P_{2,\underline{3}}$ | $T.O.L_5$ | |
| $L_5 \cup C_{\underline{n}}$ | $L_5 \cup P_{\underline{5},\underline{2}}.C'_{\underline{3}}.C'_{\underline{n-10}}$ | $T.O.L_{n-10}$ | $n \geq 13$ |
| $L_5 \cup C_{\underline{12}} \cup C_n$ | $L_5 \cup P_{\underline{5},2}.C'_{\underline{3}}.P_{\underline{4},n-2}$ | $T.O.L_{n+2}$ | $n \geq 7$ |
| $L_5 \cup C_{\underline{11}} \cup C_n$ | $L_5 \cup P_{\underline{4},3}.C'_{\underline{3}}.P_{\underline{4},n-3}$ | $T.O.L_{n+1}$ | $n \geq 7$ |
| $L_5 \cup C_{\underline{10}}$ | $L_5 \cup P_{\underline{4},\underline{3}}.C'_{\underline{3}}$ | $T.O.L_0$ | |
| $L_5 \cup C_{\underline{9}} \cup C_n$ | $L_5 \cup P_{\underline{5},2}.C'_3.P_{\underline{4},n-5}$ | $T.O.L_{n-1}$ | $n \geq 7$ |
| $L_5 \cup C_{\underline{8}} \cup C_n$ | $L_5 \cup P_{\underline{5},2}.C'_3.P_{\underline{3},n-5}$ | $T.O.L_{n-2}$ | $n = 7, 8$ |
| $L_5 \cup C_{\underline{7}} \cup C_7$ | $L_5 \cup P_{\underline{5},\underline{2}}.C'_3.C'_4$ | $T.O.L_4$ | |
| $L_4 \cup C_{\underline{n}}$ | $L_4 \cup P_{\underline{4},\underline{3},\underline{2},\underline{2}}.C'_{\underline{n-11}}$ | $T.O.L_{n-11}$ | $n \geq 14$ |
| $L_4 \cup C_{\underline{13}} \cup C_n$ | $L_4 \cup P_{\underline{4},\underline{3},\underline{2},2}.P_{\underline{4},n-2}$ | $T.O.L_{n+2}$ | $n \geq 7$ |
| $L_4 \cup C_{\underline{12}} \cup C_n$ | $L_4 \cup P_{\underline{4},3,\underline{2},\underline{2}}.P_{\underline{4},n-3}$ | $T.O.L_{n+1}$ | $n \geq 7$ |
| $L_4 \cup C_{\underline{11}}$ | $L_4 \cup P_{\underline{4},\underline{3},\underline{2},\underline{2}}$ | $T.O.L_0$ | |
| $L_4 \cup C_{\underline{10}} \cup C_n$ | $L_4 \cup P_{5,\underline{2},\underline{2},\underline{2}}.P_{\underline{4},n-5}$ | $T.O.L_{n-1}$ | $n \geq 7$ |
| $L_4 \cup C_{\underline{9}} \cup C_n$ | $L_4 \cup P_{\underline{5},2,2,2}.P_{\underline{4},n-6}$ | $T.O.L_{n-2}$ | $n \geq 8$ |
| $L_4 \cup C_{\underline{8}} \cup C_8$ | $L_4 \cup P_{\underline{5},2,2,2}.P_{\underline{3},2}$ | $T.O.L_5$ | |
| $L_4 \cup C_{\underline{7}} \cup C_n$ | $L_4 \cup P_{\underline{5},\underline{2},2,2}.C'_{n-4}$ | $T.O.L_{n-4}$ | $n \geq 7$ |

$$P_{2,2} \cup P_{4,3,2,2} \mapsto \{P(0,v,1), P(0,3,1); P(4,1,2,0,5), P(4,0,1,5), P(4,2,5), P(4,3,5)\}$$
$$P_{2,2} \cup P_{5,2,2,2} \mapsto \{P(0,v,1), P(0,3,1); P(4,0,1,2,0,5), P(4,1,5), P(4,2,5), P(4,3,5)\}$$

□

**Theorem 16.** *Suppose that either* $m + \sum m_i \geq 18$ *or* $m + \sum m_i = 15$*, with* $m \geq 0$*,* $m \neq 1, 2$ *and* $m_i \geq 7$*. For some subset* $S$ *and some* $m'$ *we can pack* $L_m \cup (\cup_{i \in S} C_{m_i})$ *into* $T.O.L_{m'}$ *exactly with initial link matching.*

**Proof.** We can assume $m > 0$ by first packing some $C_{m_{i_0}}$ into $L_{m_{i_0}}$. If $m = 3$ we pack $L_m$ into $T$ and use Theorem 14 with $m = 0$. If $m \geq 9$ we pack $L_m = P_{4,m-4}$ into $T.L_{m-3} = P_{1,2,4,m-7}$ and use Theorem 14. If $m = 6$ and $L_6 = P_{4,2}$ we pack it into $P_{1,2}.C'_3 = T.L_3$ and use Theorem 14. Since there are no $C_5$'s the result follows in each of these cases. Otherwise $4 \leq m \leq 8$ and if $m = 6$ we can assume $L_m = P_{3,3}$. In each of these remaining cases we can use one of the packings listed in Table 2. As before we can pack graph (A) into (B) by linking up suitable paths, with the initial link of $L_m$ in (A) mapped to the initial link in (B). We can then pack (B) into (C) by Lemma 15. □

**Corollary 17.**

*1 If $\sum m_i = 12$, $m_i \geq 3$ then we can pack $\cup C_{m_i}$ into $O$.*
*2 If $\sum m_i = 15$, $m_i \geq 3$ then we can pack $\cup C_{m_i}$ into $T.O$.*

**Proof.**

1. Follows from Lemma 8 since we can pack a subgraph of $K_6$ of size 12 and this subgraph must be isomorphic to $K_6' = O$.

2. If $m_j = 3$ we can pack $C_{m_j}$ into $T$ and then use part 1 to pack the remaining cycles. If $m_j \geq 6$ then we can pack $C_{m_j-3} \cup (\cup_{i \neq j} C_{m_i})$ into $O$. Since the vertices of $O$ are equivalent, we may assume $C_{m_j-3}$ meets $T$ and hence forms a circuit of length $m_j$. The only remaining case is $C_5 \cup C_5 \cup C_5 \mapsto P_{1,2}.(P_{4,3} \cup L_5) \mapsto T.O$ (using Lemma 13). □

## 9. Packing Cycles into Trails of $O$'s and $T.O$'s

Now we put the results of the previous sections together to get a proof of Theorem 1 in the cases when $N \equiv 2$ or 3 mod 4. We use the results of the last section to pack (almost) arbitrary cycles into trails of $O$'s and $T.O$'s, then we use Theorem 12 to pack these trails into some $G_{n,r}$ and then Lemma 3 to pack everything into $K_N$. The details, as usual, are somewhat more complicated.

**Theorem 18.** *Suppose $\sum m_i \geq 12a + 9$ with $m_i \geq 3$. We can pack some subset of the cycles exactly into some graph of the form*

$$O^{\cdot a} \qquad \text{or} \qquad O^{\cdot a}.P_{c,d} \qquad \text{or} \qquad O^{\cdot a-1}.P_{2,2} \cup O.T.$$

*The last form is only needed if $n_5 \geq 7$ and $a \geq 3$.*

**Proof.** By Corollary 17, we can pack $O$ with any combination of cycles of total length exactly 12. If we can pack the first $O$ in the trail in this way then we are done by induction on $a$. Hence we may assume no combination of cycles has total length 12. In particular $n_4 \leq 2$ and $n_3 + 2n_6 \leq 3$. We shall now try to pack any remaining $C_3$'s, $C_4$'s and $C_6$'s. In general, not all of these can be packed, so we may need to discard some. Provided we do not discard cycles of total length more than 6, we shall still have $\sum m_i \geq 12a + 3$. We can pack $C_4 \cup C_4 \mapsto P_{2,2,2,2}$, $C_3 \mapsto L_3$ and $C_4 \mapsto L_4$. Packing the maximum length like this will leave the remaining $C_3$'s, $C_4$'s and $C_6$'s with total length at most 6 except in the cases when $n_3 + 2n_6 = 3$ and $n_4 = 1$ or 2. In these cases we can assume $n_5 = n_8 = n_9 = 0$ (since $5 + 3 + 4 = 8 + 4 = 9 + 3 = 12$). Since the total length of all cycles is at least $12 + 9 > 17 \geq 3n_3 + 4n_4 + 6n_6$, we must have at least one cycle $C_n$ with $n = 7$ or $n \geq 10$. Pack $C_n$ into $C_3.C_{n-3}$ and $C_3 \cup C_3 \cup C_6$ or $C_3 \cup C_3 \cup C_3 \cup C_3$ into $O$. Since at least one of the $C_3$'s meets vertex number 5 of $O$, we can pack the $C_3$'s, $C_6$'s and $C_n$ into $O.C'_{n-3}$. If $n_4 = 1$ or $a = 1$ discard the remaining $C_4$'s and pack $O.C'_{n-3} \mapsto O.L_{n-3}$. Otherwise pack the $C'_{n-3}$ into $C'_4.C'_{n-7}$ (if $n \geq 10$) and $C_4 \cup C_4 \cup C_4 \mapsto O$. We can assume one of these $C_4$'s meets both vertex 1 and 5 of $O$, so we have a packing $C'_{n-3} \cup C_4 \cup C_4 \mapsto O.C'_{n-7}$ with initial links matching. We therefore obtain a packing into $O.O.L_{n-7}$ with no cycles discarded.

Now we pack the other cycles. We can assume that we have already packed an $O^{\cdot b}.L_n$ or $O^{\cdot b}.P_{2,2,2,2}$ with $0 \leq b < a$. If there are some $C_5$'s remaining, we can also assume $n \in \{0, 3, 4, 6\}$. We pack the remaining cycles inductively into graphs of the same form with larger values of $b$. If we have enough remaining $C_5$'s use the packings

$$\begin{gathered} L_3 \cup C_5 \cup C_5 \cup C_5 \mapsto O.L_6 \\ L_6 \cup C_5 \cup C_5 \mapsto O.L_4, \\ P_{2,2,2,2} \cup C_5 \cup C_5 \mapsto O.L_6 \\ C_5 \cup C_5 \cup C_5 \cup C_5 \mapsto P_\emptyset \cup P_{3,3,3,3}.P_{2,2,2,2} \mapsto O.P_{2,2,2,2}, \\ L_4 \cup C_5 \cup C_5 \cup C_5 \cup C_5 \mapsto P_{2,2} \cup P_{2,2,2,2}.P_{3,3,3,3} \mapsto O.O. \end{gathered}$$

The first three of these packings come from Theorem 14, the last two follow from Lemma 13. In each case the initial links match, so we can pack $O^{\cdot b}.L_n$ or $O^{\cdot b}.P_{2,2,2,2}$ into $O^{\cdot b+1}.L_{n'}$ or $O^{\cdot b+1}.P_{2,2,2,2}$ or $O^{\cdot b+2} = O^{\cdot b+2}.L_0$. When $b+1 = a$, we must avoid the last two forms if we are to pack graphs of the type listed in the statement of the theorem.

Assume we have enough $C_5$'s to reach a total length of at least $12a+3$. We shall use up all the $O$'s except in the cases when we have packed $O^{\cdot a-1}.L_n$ with $n = 0$ or 4. If $n = 0$ we can use

$$C_5 \cup C_5 \cup C_5 \mapsto L_5 \cup P_{4,3}.P_{1,2} \mapsto O.P_{1,2}.$$

However, for $n = 4$ we cannot pack $L_4 \cup C_5 \cup C_5 \cup C_5$ into the final $O$. If the $L_4$ came from an original $C_4$, we can discard the $C_4$ and pack $C_5 \cup C_5 \cup C_5 \mapsto O.P_{1,2}$ for the last $O$. Otherwise we must have had at least seven $C_5$'s and three $O$'s (the minimum number occurs if we started with $P_{2,2,2,2}$ in the first $O$). For this case, pack $C_5 \cup C_5 \cup C_5 \mapsto O.T$. We then get a graph of the form $O^{\cdot a-1}.L_4 \cup O.T$.

Now assume we do not have enough $C_5$'s to pack all the octahedra. After packing as many $C_5$'s as possible, we shall have at most two $C_5$'s left or three $C_5$'s if we have packed $O^{\cdot b}.L_0$ or $O^{\cdot b}.L_4$. Use the $L_0 \cup C_5 \mapsto L_5$ and $L_4 \cup C_5 \cup C_n \mapsto O.L_{n-3}$ packings from Theorem 14 to ensure we have at most two $C_5$'s left. Now continue packing the remaining cycles using Theorem 14. Whenever we have packed $O^{\cdot b}.L_n$ or $O^{\cdot b}.P_{2,2,2,2}$ with $b < a$, we have enough extra cycles to pack $L_n$ or $P_{2,2,2,2}$ and some cycles into $O.L_{n'}$ with initial link matching by Theorem 14, and hence we can pack $O^{\cdot b+1}.L_{n'}$. The only exception is when we try to pack $L_9 \cup C_5 \cup C_5$ into the last $O$ (since there are at most two $C_5$'s and the other combinations not allowed by Theorem 14 have too few edges). Since $9+5+5 < 12+9$, even this case does not occur unless there are some discarded cycles. Since $n_5 \neq 0$, we have $n_3 n_4 = 0$ and the only possible discarded cycles are $C_3$'s and $C_6$'s. In this case use the discarded cycles with the packings

$$L_9 \cup C_3 \mapsto L_9 \cup L_3 \mapsto O.L_0 \quad \text{and} \quad L_9 \cup C_6 \mapsto L_9 \cup L_3.C'_3 \mapsto O.P_{1,2}.$$

In the general case we finish by packing the last $L_n$ as $P_{1,2}$ if $n = 3$. □

**Theorem 19.** *Suppose $L = \sum m_i \geq 12a + 15m + 11$, with $m_i \geq 3$, $m$ even and $L - 4n_4 \geq 15m + 3$. Then we can pack some subset of the cycles exactly into some graph of the form*

$$O^{\cup a-b} \cup (T.O)^{\cdot m}.O^{\cdot b}, \qquad O^{\cup a-b} \cup (T.O)^{\cdot m}.O^{\cdot b}.P_{c,d}$$
$$\text{or} \qquad O^{\cup a-b} \cup (T.O)^{\cdot m}.O^{\cdot b-1}.P_{2,2} \cup O.T,$$

*where the non-link vertices of the $T$'s in $(T.O)^{\cdot m}$ are identified. The last form only occurs with $a \geq b \geq 3$.*

**Proof.** By discarding $C_4$'s we may assume either $n_4 = 0$ or $L \leq 12a + 15m + 14$. In either case $n_4 < 3a + 3$. We set $b = a - \lfloor n_4/3 \rfloor \geq 0$ and pack $C_4$'s into the isolated $O$'s of $O^{\cup a-b}$ using $C_4 \cup C_4 \cup C_4 \mapsto O$. Therefore, without loss of generality, we may assume $n_4 \leq 2$ and set $b = a$.

By Corollary 17 we can pack $C_5 \cup C_5 \cup C_5 \mapsto T.O$. Doing this twice and using induction on $m$ we can assume either $m = 0$ or $n_5 \leq 5$. The case $m = 0$ follows immediately from Theorem 18. Since the $T$'s in the $(T.O)^{\cdot m}$ are all linked via a common vertex, we can pack these triangles with $C_6$'s, and, if necessary, by $C_3$'s. If we run out of $T$'s, then we must have exactly used them all ($m$ is even). We are then done by Theorem 18, noting that $n_5 \leq 5$ so the last form in Theorem 18 will not occur.

If we run out of $C_3$'s and $C_6$'s, drop the assumption that $m$ is even and assume $n_3 = n_6 = 0$. By using $C_5 \cup C_5 \cup C_5 \mapsto T.O$ again we may also assume either $m = 0$ or $n_5 \leq 2$. If $m = 0$ we are done by Theorem 18 again. Discarding any $C_4$'s (of total length at most 8) we may now assume $\sum m_i \geq 12a + 15m + 3$ with $n_3 = n_4 = n_6 = 0$, $n_5 \leq 2$ and without the condition that $m$ is even.

We can pack $C_5 \cup C_n \mapsto P_{1,2}.P_{4,n-2} = T.L_{n+2}$ for $n \geq 5$, so that we now have no $C_5$'s left (choose $n = 5$ if $n_5 = 2$ and $n \geq 7$ if $n_5 = 1$. There must be at least one cycle of length at least 7 since $\sum_{j<7} jn_j \leq 5 + 5 < 15 + 3$). Now inductively pack the remaining cycles $C_{m_i}$, $m_i \geq 7$ according to Theorem 16 and Theorem 14 in a manner analogous to the proof of Theorem 18. □

**Theorem 20.** *Assume the Induction Hypothesis with $N > 12$, $N \equiv 2$ or $3 \bmod 4$. Then Theorem 1 holds for $N$.*

**Proof.**

1. The case $N \equiv 2 \bmod 4$, $N \geq 14$.

By part 1 of Theorem 12, there are exact packings of $O^{\cdot a}$, and $O \cup O^{\cdot a-1}$ into some $G_{4n,2r}$, $4n + 2r = N$, $2r \geq 6$, with the final link in $R$. The isolated $O$ in $O \cup O^{\cdot a-1}$ can be required to have its final link in $R$ and disjoint from the final link of $O^{\cdot a-1}$. Now $L \geq |E(K'_N)| - 2 = |E(G_{4n,2r})| + |E(K'_{2r})| - 2 \geq 12a + 10$. Using Theorem 18 we can pack some cycles into $O^{\cdot a}$, $O^{\cdot a}.P_{c,d}$ or $O.T \cup O^{\cdot a-1}.P_{2,2}$. These can then be packed into $G_{4n,2r}.(\cup C_{l_i})$ with at most two attached cycles $C_{l_i}$ meeting $R$ in two vertices each. The cycles $C_{l_i}$ are just the $P_{c,d}$ or the $T$ and $P_{2,2}$ linked to the ends of the trails of octahedra. If there are two such cycles then one is a triangle ($T$) and their intersections with $R$ are disjoint. Therefore we are done by either part 4 or part 5 of Lemma 3.

2. The case $N \equiv 3 \bmod 4$, $N \geq 15$.
The proof is similar, using part 2 of Theorem 12 and Theorem 19. We have $4n+2r+1 = N$, $m = 2n$, $n \geq 2$, $r \geq 3$ and $L = |E(G_{4n,2r+1})| + |E(K_{2r+1})| \geq 12a + 15m + 21$. In the case $N = 15$ we also use the fact that $a = 2$ in Theorem 19 to avoid getting a packing of the last type. The result follows whenever $L - 4n_4 \geq 15m+3 = 30n+3$. By Lemma 4 we may assume $n_4 < \frac{1}{2}(N-3)$, so $4n_4 \leq 2N-10$. It can be checked that $\binom{N}{2} - (2N-10) \geq 30n+3$ for $N \geq 15$, so the result follows. □

## 10. Packing Cycles into Linked $K_5$'s.

In this section we prove Theorem 1 for $N \equiv 1 \bmod 4$. The idea is the same as in the previous section, but we also need to be able to pack (almost) arbitrary cycles into linked $K_5$'s.

**Lemma 21.** *Assume $a \geq 2$, $m_i \geq 3$ and $\sum m_i - \max(3n_3+6n_6+9n_9, 7n_7) - 4n_4 - 8n_8 \geq 10a + 3$. Then we can pack some subset of the cycles exactly into a graph of the form*

$$K_5^{\cdot a} \qquad or \qquad K_5^{\cdot a}.C_m \qquad or \qquad C_5.K_5^{\cdot a}.C_m,$$

*where the $K_5$'s are all linked at a common vertex $v$ and the extra cycles are linked to distinct $K_5$'s at single vertices which may be chosen arbitrarily (either or both possibly equal to $v$).*

**Proof.** We first prove the result with the extra cycles linked at $v$. Since Theorem 1 holds for $N = 5$, we can pack any collection of cycles of total length 10 into a $K_5$. If we run out of $K_5$'s we are done. We can pack $C_3 \cup C_7$ into $K_5$ so that the $C_3$ meets $v$. Since all the $K_5$'s are linked at $v$, if we remove $C_7$'s from the $K_5$'s we get triangles linked at a common vertex $v$. Therefore we can pack $C_6$'s and $C_9$'s with $C_7$'s into $K_5$'s rather than just $C_3$'s. Since we use $C_6$'s and $C_9$'s to pack the triangles, we may need to link an extra $C_3$ or $C_6$ to $v$ to pack the last $C_6$ or $C_9$ if we run out of $C_7$'s or $K_5$'s. We may also link $C_5$ to $v$ to pack any remaining $C_5$ (we can assume there is no more than one $C_5$ since two $C_5$'s will pack into $K_5$). Assume we have not used all the $K_5$'s but we have packed a graph of one of the forms $K_5^{\cdot b}$, $K_5^{\cdot b}.C_m$ or $C_5.K_5^{\cdot b}.C_m$ with $0 \leq b < a$ and the cycles linked at $v$. Pack cycles $C_n$ with $n \geq 11$ inductively as follows:

$$\begin{array}{lll} (A) & K_5^{\cdot b} \cup C_n \mapsto K_5^{\cdot b}.C_n & \\ (B) & K_5^{\cdot b}.C_m \mapsto C_5.K_5^{\cdot b}.C_{m-5} & \text{if } m \geq 8, \\ (C) & K_5^{\cdot b}.C_m \cup C_n \mapsto K_5^{\cdot b+1}.C_{n+m-10} & \text{if } m \leq 7, \\ (D) & C_5.K_5^{\cdot b}.C_m \mapsto K_5^{\cdot b+1}.C_{m-5} & \text{if } m \geq 8, \\ (E) & C_5.K_5^{\cdot b}.C_m \cup C_n \mapsto C_5.K_5^{\cdot b+1}.C_{n+m-10} & \text{if } m \leq 7. \end{array}$$

In each case, we link the cycles at the vertex $v$. In $(B)$ and $(D)$ we have split $C_m$ as $C_5.C_{m-5}$ and packed the two $C_5$'s in $(D)$ into a $K_5$. In $(C)$ and $(E)$ we have split $C_n$ as $C_{10-m}.C_{n+m-10}$ and packed the $C_m$ and $C_{10-m}$ into $K_5$. In each of these cases, we can ensure both packed cycles meet $v$. Note that $(A)$ is used at most once at the beginning of the process and we can assume that we finish with one of $(C)$, $(D)$ or $(E)$. The operations above can pack all the $K_5$'s provided the sum of the lengths of the cycles that can be packed is at least $10a + 3$. We now prove that we shall not run out of such cycles. If $n_3 + 2n_6 + 3n_9 \geq n_7$, then we can assume that we have run out of $C_7$'s in the first part of this proof. The only remaining cycles that we can't pack into $K_5$'s are $C_n$'s with $n = 3, 6, 9, 4, 8$. Similarly if $n_3 + 2n_6 + 3n_9 \leq n_7$, then we can assume we have run out of $C_3$'s, $C_6$'s and $C_9$'s. The only cycles that we can't pack into $K_5$'s are $C_n$'s with $n = 7, 4, 8$. In both cases, the total length of the others is at least $10a + 3$ and we are done.

We now deal with the requirement that we may have to link the cycles at different vertices. If we have just one cycle $C_m$ linked to $K_5^{\cdot a}$ then it will either be one of the original cycles (in which case we can link it anywhere), or it is part of one of the original cycles which is split as $C_n.C_m$ with the $C_n$ packed inside $K_5^{\cdot a}$. In this case, we can link $C_m$ to any of the vertices of the image of $C_n$ in $K_5^{\cdot a}$. In particular, $C_m$ can be linked to some vertex other than $v$. By permuting the $K_5$'s and the vertices of $K_5 \setminus \{v\}$ we can therefore link $C_m$ to any desired vertex. The same arguments apply when we have two cycles $C_5$ and $C_m$, except that in addition we have to ensure that if both are part of larger cycles, then the other parts of these two larger cycles are packed into $K_5^{\cdot a}$ in such a way that they meet at least two distinct $K_5$'s at vertices other than $v$. For this to fail, we must have finished our packing above with operation $(E)$ in which the $C_5$ on the left hand side is not part of a cycle partially packed into the $K_5$'s. (Recall that we never finish with operation $(B)$.) If the $C_5$ was an original cycle $C_{m_i}$ then we can link $C_5$ anywhere and we are done. We may therefore assume that the $C_5$ and $C_m$ of $(E)$ came from splitting a $C_{5+m}$ which was one of the original cycles $C_{m_i}$. This means that we used operation $(B)$ with the $C_m$ on the left hand side equal to an original $C_{m_i}$. This only happens if we previously used $(A)$, and so all but one of the $K_5$'s were packed exactly at the beginning of this proof. We may assume all the remaining cycles are of lengths 11 or 12 (any cycle of length at least 13 could be packed using $(A)$, $(B)$ and $(D)$) and there are at least two of them (to get a total length of at least $10a + 3$). Hence we may assume that $K_5^{\cdot a-1}$ has been packed exactly and there are two remaining cycles $C_{m_1}$ and $C_{m_2}$ with $11 \leq m_1, m_2 \leq 12$. Since $a \geq 2$, there is at least one $K_5$ that has been packed already.

If we have packed some $K_5$ with a $C_{10}$, then we can unpack this $K_5$ and pack the $C_{10}$ and $C_{m_1}$ into $C_5.K_5.C_{m_1-5}$ as $(C_5.C_5).(C_5.C_{m_1-5}) \mapsto C_5.K_5.C_{m_1-5}$. Now proceed as above and use $C_{m_2}$ and operation $(E)$ to pack the remaining $K_5$. This time the two linked cycles are part of larger cycles meeting two $K_5$'s between them and we are done.

If no $K_5$ was packed with a $C_{10}$, pick one packed $K_5$ and remove all except one of the original cycles, $C_{m_3}$ say, that was packed entirely within it. If we choose $m_3$ to be minimal then we can assume $m_3 \leq 5$. Pack $C_{m_3}$ and a length $10 - m_3$ of $C_{m_1}$ into this $K_5$. We have now packed $K_5^{\cdot a-1}.C_{m_1+m_3-10}$ with $4 \leq m_1 + m_3 - 10 \leq 7$, and have $C_{m_2}$ left to pack. We proceed by packing $C_{m_2}$ using operation $(C)$. This time we have only one cycle linked to $K_5^{\cdot a}$ and we are done. □

The reason we need to link the cycles in Lemma 21 to arbitrary vertices is because we shall pack the linked cycles $C_5$ and $C_m$ into the trail of octahedra and hence may wish to link them to $K_5^{\cdot a}$ at vertices in $M$. We need to do this for two reasons. One is that we cannot have two cycles linked at the same vertex $v$ of $R$ in Lemma 3. The other is that we have no control over the size of $C_m$. Indeed, $C_m$ may be so large that there are not enough remaining cycles to pack the trail of octahedra completely.

**Theorem 22.** *Assume the Induction Hypothesis with $N > 12$, $N \equiv 1 \bmod 4$. Then Theorem 1 holds for $N$.*

**Proof.** We use the construction in Theorem 12 to pack $K_5^{\cdot n} \cup O^{\cdot a}$ or $K_5^{\cdot n} \cup O \cup O^{\cdot a-1}$ into $G_{4n,2r+1}$ where $4n+2r+1 = N$, $n \geq 2$ and $2r+1 \geq 5$. By Lemma 4 we can assume $4n_4 + 8n_8 \leq 2N - 10$ and by Lemma 5 we can assume $\max(3n_3 + 6n_6 + 9n_9, 7n_7) \leq \frac{7}{2}(N-3)$. It can be checked that $\binom{N}{2} - \frac{7}{2}(N-3) - (2N-10) \geq 10n + 3$, so we can use Lemma 21 to pack some cycles into $K_5^{\cdot n}$ with at most two extra cycles $C_{l_i}$ attached. Since $\binom{N}{2} - 10n = |E(G_{4n,2r+1})| - 10n + |E(K_{2r+1})| \geq 12a + 10$, we can use Theorem 18 to pack some subset of the $C_{l_i}$ and the remaining $C_{m_i}$ into $O^{\cdot a}$, $O^{\cdot a}.P_{c,d}$ or $O^{\cdot a-1}.P_{2,2} \cup O.T$. This last form occurs only when there are at least seven $C_5$'s, so by the algorithm of Lemma 21 we can assume there are no $C_{l_i}$ attached to $K_5^{\cdot n}$ in this case (we use up all except at most one of the $C_5$'s before linking any cycles to $K_5^{\cdot n}$). If there are no $C_{l_i}$, we are done by the same proof as in Theorem 20. Assume now that there is precisely one cycle $C_m = C_{l_1}$ linked to $K_5^{\cdot n}$. If this cycle was packed in the $O^{\cdot a}$, it must have at least one vertex in $M$ when packed into $G_{4n,2r+1}$. It therefore meets some $K_5$, and so we can assume that this cycle is linked to the correct vertex by Lemma 21. (Recall from the proof of Theorem 12 that every vertex of $M$ meets precisely one $K_5$.) If the cycle was not packed, link it at $v$. We now have at most one cycle linked at $v$ and possibly one cycle $P_{c,d}$ linked at two other vertices of $R$. The result now follows using part 1 or 4 of Lemma 3 with $k \leq 2$.

Now assume there are two cycles $C_5$ and $C_m$ which must be linked to $K_5^{\cdot n}$. If precisely one of $C_5$ and $C_m$ is not packed into the trail of octahedra, we link it to $v$ and link the other to a vertex in $M$ as above. If both are packed and the $C_5$ has an edge in $M$, then this $C_5$ must meet two $K_5$'s and so we can link the $C_m$ and $C_5$ to distinct $K_5$'s via vertices in $M$. If $C_5$ does not have an edge in $M$ then it must have an even number of edges in $G_{4n,2r+1}$ since this graph has no edges in $R$. Therefore the $C_5$ must be partially packed with a path of odd length leaving the $O$-trail. We can assume this path is of length 3, since the only case where we have a pentagon with one edge in $R$ is when we are packing three of them as $O.T$. In this case, permute the $C_5$'s and assume our $C_5$ is entirely within the $O$. In the case when the $C_5$ has a path of length 3 in $R$, we can permute the vertices of $R$ in any final packing so that this path meets $v$ and we are done. (We can permute all the vertices of $R$ except the two link vertices of the cycle $P_{c,d}$ that may be linked to the trail of octahedra.)

Finally we need to deal with the case when neither $C_5$ nor $C_m$ is packed at all in $O^{\cdot a}$. If $r = 2$, then these cycles can be packed together with any remaining cycles exactly into $K_R = K_5$. Hence $m = 5$ and both $C_5$ and $C_m$ meet $v$. In all other cases $r \geq 4$ and $|E(K_R)| \geq |E(K_9)| = 36$. Therefore when using Theorem 18, we have $\sum m_i \geq 12a + 36$. Looking at the proof of Theorem 18, we see that, in general, we can discard

$C_3$'s, $C_4$'s and $C_6$'s so as to make the algorithm pack a $C_5$ as $L_5$ before packing the longer cycles. This is because the maximum length of discarded cycles will be at most 17 and $12a+36-17 \geq 12a+9$. So if our $C_5$ could not be packed, we must have already packed all the $O$'s exactly with combinations of cycles with lengths totaling 12 at the beginning of the proof of Theorem 18. Pick the last $O$ packed. We can assume it is not packed with any cycle of length 5, $m$, 7, $12-m$ or $7-m$, since in these cases we can either swap $C_5$ or $C_m$ with this packed cycle or swap the other cycles packed into this $O$ with $C_5$ and/or $C_m$. We can also assume it is not packed with any cycle $C_r$ with $r \geq 8$, since we could replace the $C_r$ with $C_{r-5}$ and our $C_5$ and get a packing of $O.C_5' \mapsto O.L_5$ with our $C_5$ inside the $O$. Hence, without loss of generality, this $O$ is packed as $C_3 \cup C_3 \cup C_3 \cup C_3$, $C_3 \cup C_3 \cup C_6$, $C_6 \cup C_6$ or $C_4 \cup C_4 \cup C_4$ and $C_m$ is of length $m \geq 5$ in all except the $C_6 \cup C_6$ case. We now use the following packings in which the $C_5$ is used to pack this last $O$:

$$C_3 \cup C_3 \cup C_5 \cup C_m \mapsto C_3' \cup C_3'.P_{3,3}.P_{2,m-3} \mapsto O.P_{2,m-3},$$
$$C_6 \cup C_5 \cup C_6 \mapsto L_6 \cup P_{3,3}.P_{2,3} \mapsto O.P_{2,3},$$
$$C_4 \cup C_4 \cup C_5 \cup C_m \mapsto P_{2,2,2,2} \cup P_{2,2}.P_{3,m-2} \mapsto O.P_{3,m-2}.$$

We can now assume the $C_5$ is packed within the octahedra and so we are done. □

## 11. Packing Cycles into Linked $W$'s.

In this section we complete the proof of Theorem 1 by handling the cases $N \equiv 0 \bmod 4$. In this case we shall pack cycles into linked $W$'s where $W = G_{4,4} = C_4 + E_4$ and the vertices of the $E_4$ are both the initial and final links of $W$ (in the same order). Let these link vertices of $W$ be $(a,b,c,d)$. We shall use notations such as $[l_1^a l_2^b l_3^{bc} \dots]$ to denote the vertices $\{a,b,c,d\}$ and a collection of cycles $C_{l_i}$ with with $C_{l_1}$ meeting $a$, $C_{l_2}$ meeting $b$, $C_{l_3}$ meeting both $b$ and $c$, etc.. The initial and final links of this graph will be $(a,b,c,d)$. If a cycle meets more than one vertex this notation can be ambiguous, so to be precise, say that the vertices occur at intervals two apart around the cycle. Define $\mathcal{D}$ to be the family of graphs

$$[\,],\quad [l_1^u],\quad [l_1^u l_2^v] \text{ with } l_2 \leq 5,\quad [3^u 3^v 3^w],\quad [4^u 4^v 4^w]\quad \text{or}\quad [4^a 4^b 4^c 4^d],$$

where $u$, $v$ and $w$ are distinct elements of $\{a,b,c,d\}$.

**Lemma 23.** *We can pack any of the following exactly into $W$ with link matching:*

$$[l_1^a l_2^b l_3^c l_4^d],\quad [l_1^a l_2^b l_3^c],\quad [l_1^a l_2^b],\quad [l_1^a] \qquad \textit{with } \textstyle\sum l_i = 20,$$
$$[3^a 3^b 3^c 8^d 3^a],\quad [3^a 3^b 3^c 6^d 5^a],\quad [4^a 4^b 4^c 4^d 4^a] \quad \textit{and} \quad [4^a 4^b 4^c 5^d 3^a].$$

**Proof.** Write the non-link vertices of $W = C_4 + E_4$ as $C_4 = C(1,2,3,4)$. We have the following packings into $W$:

| | | |
|---|---|---|
| (A) | $[3^a 3^a \underline{3}^b \underline{3}^c 4^{bd} 4^{cd}]$ | $C(a,1,2), C(a,3,4), \underline{C(b,2,3)}, \underline{C(c,1,4)}, C(b,1,d,4), C(c,2,d,3)$ |
| (B) | $[\underline{3}^a 3^a \underline{4}^{bd} 5^{bc} 5^{cd}]$ | $\underline{C(a,1,2)}, C(a,3,4), \underline{C(b,3,d,4)}, C(b,1,c,3,2), C(c,2,d,1,4)$ |
| (C) | $[5^{ab} 5^{bd} 5^{ac} 5^{cd}]$ | $C(a,3,b,2,1), C(b,1,d,3,4), C(a,4,c,3,2), C(c,2,d,4,1)$ |
| (D) | $[\underline{3}^a 3^b \underline{3}^c 5^{ad} 6^{bcd}]$ | $\underline{C(a,1,2)}, C(b,4,1), \underline{C(c,3,4)}, C(a,4,d,2,3), C(b,2,c,1,d,3)$ |
| (E) | $[4^{ad} 4^{bc} 4^{bc} 5^d 3^a]$ | $C(a,2,d,3), C(b,1,c,3), C(b,2,c,4), C(d,1,2,3,4), C(a,1,4)$ |
| (F) | $[4^a 4^b 4^{cd} 4^{cd} 4^{ab}]$ | $C(a,1,2,3), C(b,3,4,1), C(c,1,d,2), C(c,3,d,4), C(a,2,b,4)$ |

In each of these packings every cycle meets every other, except in (A), (B) and (D) where the two underlined cycles do not meet. We now combine cycles to give some of the packings in the statement of the lemma. In each case we indicate one of the above packings that can be used.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $[3^a \underline{3}^b \underline{3}^c 11^d]$ | $(A)$ | $[3^a 3^b 7^c 7^d]$ | $(A)$ | $[3^a 5^b 5^c 7^d]$ | $(B)$ | $[4^c 4^d 6^a 6^b]$ | $(A)$ |
| $[3^a 3^b 4^c 10^d]$ | $(A)$ | $[3^b 4^c 4^d 9^a]$ | $(A)$ | $[3^b 5^a 6^c 6^d]$ | $(D)$ | $[4^d 5^b 5^c 6^a]$ | $(B)$ |
| $[3^b 3^c 5^d 9^a]$ | $(D)$ | $[3^a 4^b 5^d 8^c]$ | $(E)$ | $[4^a 4^b 4^c 8^d]$ | $(E)$ | $[5^a 5^b 5^c 5^d]$ | $(C)$ |
| $[3^a 3^b 6^c 8^d]$ | $(A)$ | $[3^a 4^b 6^c 7^d]$ | $(A)$ | $[4^a 4^b 5^d 7^c]$ | $(E)$ | $[3^a \underline{3}^b \underline{3}^c 8^d 3^a]$ | $(A)$ |

In all the cases in the table above, every cycle meets every other one in the packing except for the underlined ones in the first and last cases. Together with (D), (E) and (F), these give all the cases in the statement of the lemma with at least four cycles. (We can clearly permute the link vertices by symmetry.) If we have fewer than four cycles, one of these cycles must have length at least 6, and we can split it into two smaller cycles. These cases therefore follow by combining cycles in the above table that are connected. We can always avoid combining underlined cycles by using different triangles. □

**Theorem 24.** *If $\sum m_i \geq 20n + 29$ and $n_3 = n_6 = 0$ then there is an exact packing of some subset of the cycles into a graph $W^{\cdot n}.D$ for some $D \in \mathcal{D}$.*

**Proof.** By packing four $C_5$'s into a $W$ (using the $[5^a 5^b 5^c 5^d]$ packing of Lemma 23), we can assume by induction on $n$ that $n_5 \leq 3$. If $n_5 \geq 2$ and $n_7 \geq 2$ we can pack two $C_7$'s and two $C_5$'s as $[7^a 5^b 5^c 7^d] \mapsto [3^a 5^b 5^c 7^d].[4^a] \mapsto W.[4^a]$. Pack $C_4$'s by adding them to each link vertex in turn to get $[4^a]$, $[4^a 4^b]$, $[4^a 4^b 4^c]$ and $[4^a 4^b 4^c 4^d]$. If we have an extra $C_4$ we can pack this into a $W$ using $[4^a 4^b 4^c 4^d 4^a] \mapsto W$. Continue until we run out of $C_4$'s. We can now assume we have packed $W^{\cdot r}.D$ with $r < n$ and $D = []$, $[4^a]$, $[4^a 4^b]$, $[4^a 4^b 4^c]$ or $[4^a 4^b 4^c 4^d]$. Now pack $C_7$'s inductively using the following packings:

$$
\begin{array}{lclcl}
[].[7^a 7^b 7^c 7^d] & \mapsto & [3^a 3^b 7^c 7^d].[4^a 4^b] & \mapsto & W.[4^a 4^b] \\
[4^a].[7^a 7^b 7^c 7^d] & \mapsto & [3^b 3^c 7^a 7^d].[4^a 4^b 4^c] & \mapsto & W.[4^a 4^b 4^c] \\
[4^a 4^b].[7^a 7^c 7^d] & \mapsto & [4^a 4^b 4^c 4^d 4^a].[3^a 3^c 3^d] & \mapsto & W.[3^a 3^c 3^d] \\
[4^a 4^b 4^c].[7^a 7^d] & \mapsto & [4^a 4^b 4^c 4^d 4^a].[3^a 3^d] & \mapsto & W.[3^a 3^d] \\
[4^a 4^b 4^c 4^d].[7^a] & \mapsto & [4^a 4^b 4^c 4^d 4^a].[3^a] & \mapsto & W.[3^a] \\
[3^a].[7^b 7^c 7^d] & \mapsto & [3^a 3^b 7^c 7^d].[4^b] & \mapsto & W.[4^b] \\
[3^a 3^d].[7^b 7^c] & \mapsto & [3^a 3^d 7^b 7^c] & \mapsto & W \\
[3^a 3^c 3^d].[7^a 7^b] & \mapsto & [3^c 3^d 7^a 7^b].[3^a] & \mapsto & W.[3^a]
\end{array}
$$

(We may need to permute the link vertices at some points.) Now discard any remaining $C_5$'s and $C_7$'s. The maximum length of discarded cycles is $\max(5 + 3 \times 7, 3 \times 5 + 7) = 26$.

Hence, we can assume that the total length of cycles is at least $20n+3$, all the remaining cycles are of length at least eight and we have already packed some $W^{\cdot r}.D$ with $r < n$ and $D \in \mathcal{D}$. We now proceed inductively in a manner similar to before using Lemma 23. At each stage add cycles to any unused link vertices until the total length to be packed is at least 23 or all four link vertices have been used.

Suppose first that we have cycles of total length $s \geq 23$ linked to the link vertices. If the longest cycle is $l_1$ and $s - l_1 \leq 17$, then we can split $C_{l_1}$ as $C_{l_1+20-s}.C_{s-20}$ (both linked to the same link vertex $v$ as $C_{l_1}$). Now pack the $C_{l_1+20-s}$ and the remaining cycles into $W$ using Lemma 23. We now have a packing of $W^{\cdot r+1}.[(s-20)^v] = W^{\cdot r+1}.D'$, where $D' \in \mathcal{D}$. If $s-l_1 \geq 18$ then there must be another linked cycle, $C_{l_2}$ say, with $l_2 \geq 6$ (since there are at most three remaining linked cycles). Also $l_1 \geq 8$ (since it must be one of the original $C_{m_i}$). If $s - l_1 \leq 20$ split $C_{l_2}$ as $C_{l_2-3}.C_3$ and split $C_{l_1}$ as as $C_{l_1+23-s}.C_{s-23}$. This time use Lemma 23 to obtain a packing into $W^{\cdot r+1}.[(s-23)^v 3^u] = W^{\cdot r+1}.D'$. If $s - l_1 = 21$ then we may assume $l_2 \geq 7$ and if $s - l_2 = 22$ we may assume $l_2 \geq 8$. We can treat these similarly and get packings into $W^{\cdot r+1}.[(s-24)^v 4^u]$ and $W^{\cdot r+1}.[(s-25)^v 5^u]$ respectively. Hence in all cases we get a packing into $W^{\cdot r+1}.D'$ for some $D' \in \mathcal{D}$.

The remaining cases are when we run out of link vertices. For this to happen, we must have cycles attached as $[3^a 3^b 8^c 8^d]$, $[3^a 3^b 3^c l^d]$ with $8 \leq l \leq 13$ or $[4^a 4^b 4^c l^d]$ with $l = 4$, 9 or 10. We must also have another cycle $C_{m_1}$, say, with $m_1 \geq 8$. Now use the following packings:

$$\begin{array}{lcll}
[3^a 3^b 8^c 8^d].[m_1^a] & \mapsto & [3^a 3^b 3^c 8^d 3^a].[(m_1-3)^a 5^c] & \\
[3^a 3^b 3^c l^d].[m_1^a] & \mapsto & [3^a 3^b 3^c 8^d 3^a].[(m_1-3)^a (l-8)^d] & l = 8, 12, 13 \\
[3^a 3^b 3^c l^d].[m_1^a] & \mapsto & [3^a 3^b 3^c 6^d 5^a].[(m_1-5)^a (l-6)^d] & l = 9, 10, 11 \\
[4^a 4^b 4^c 4^d].[m_1^a] & \mapsto & [4^a 4^b 4^c 4^d 4^a].[(m_1-4)^a] & \\
[4^a 4^b 4^c l^d].[m_1^a] & \mapsto & [4^a 4^b 4^c 5^d 3^a].[(m_1-3)^a (l-5)^d] & l = 9, 10
\end{array}$$

In each case we obtain a packing into some graph of the form $W^{\cdot r+1}.D'$, $D' \in \mathcal{D}$, using one of the packings $(A)$-$(F)$ from the proof of Lemma 23.

Hence by induction we can pack $W^{\cdot n}.D$ for some $D \in \mathcal{D}$. □

**Theorem 25.** *Assume the Induction Hypothesis with $N > 12$, $N \equiv 0 \bmod 4$. Then Theorem 1 holds whenever $n_3 + 2n_6 \leq T_W$. Here $T_W = \lfloor (\binom{N}{2} - \frac{N}{2} - 20n - 31)/3 \rfloor$ and $n$ is given in part 4 of Theorem 12.*

**Proof.** We use the construction in Theorem 12 to pack $W^{\cdot n} \cup O^{\cup a-b} \cup O^{\cdot b}$ into $G_{4n,2r+4}$ where $4n + 2r + 4 = N$, $2r + 4 \geq 8$ and $r = r(\frac{1}{2}(N-4))$. Since $L \geq \binom{N}{2} - \frac{N}{2} - 2$ and $n_3 + 2n_6 \leq T_W$, we have $L - 3n_3 - 6n_6 \geq 20n + 29$. Hence we can use Theorem 24 to pack some cycles into $W^{\cdot n}$ with at most four extra cycles $C_{l_i}$ attached (in the form of some $D \in \mathcal{D}$). Let the largest such cycle be $C_{l_1}$ (if such cycles exists). From the definition of $\mathcal{D}$ we see that the remaining extra cycles have total length at most 12. Since $L - 20n - 12 \geq 12a + |E(K'_{2r+4})| - 14 \geq 12a + 10$, we can use Theorem 18 to pack some subset of $C_{l_1}$ and the remaining $C_{m_i}$ into $O^{\cdot a}$, $O^{\cdot a}.P_{c,d}$ or $O^{\cdot a-1}.P_{2,2} \cup O.T$. From the algorithm of Theorem 24 we can assume this last form occurs only when there are no $C_{l_i}$ (since all the $W$'s would be packed with $C_5$'s).

If there are no extra cycles $C_{l_i}$ then we are done by the proof of Theorem 20. If $C_{l_1}$ is not packed into the trail of octahedra then we are done by part 1 or part 4 of Lemma 3 (depending on whether we have packed $O^{\cdot a}$ or $O^{\cdot a}.P_{c,d}$). Now assume that $C_{l_1}$ has been packed into the octahedra. When this trail is packed into $G_{4n,2r+4}$ the cycle $C_{l_1}$ will meet $M$ (since there are no edges in $R$). If the $C_{l_1}$ was part of the larger original cycle $C_{m_1}$, say, then the other part, $C_k$ say, meets some non-link vertex of some $W$ when $C_{m_1}$ is packed in $W^{\cdot n}.(\cup C_{l_i})$. Every vertex of $M$ meets some $W$ when $W^{\cdot n}$ is packed into $G_{4n,2r+4}$. Hence, by permuting the $W$'s and cyclically permuting the non-link vertices of some $W$, we can assume that the image of $C_{l_1}$ in the trail of octahedra meets the image of $C_k$ in $W^{\cdot n}$ when everything is packed into $G_{4n,2r+4}$. Hence we have a packing of some subset of cycles into $G_{4n,2r+4}.(\cup C_{l'_i})$ satisfying one of the conditions of Lemma 3. The result now follows. □

We have now proved Theorem 1 for $N \equiv 0 \bmod 4$ when there are not too many cycles of lengths 3 or 6. However if we have many cycles of lengths 3 or 6 we will need to use other packings. For large $N$, the packings of Section 6 are sufficient. However, there are a number of smaller $N$ for which this is not sufficient. For these cases we need to give special packings. These special packings are given by the next two lemmas.

**Lemma 26.** *For all $n \geq 1$ there exists exact packings*

$$\begin{aligned} K'_{4n} \cup K'_{4n} \cup O^{\cdot 4n^2} &\quad\mapsto\quad G_{8n,4n} \\ G_{4n,4}.G_{4n,4} \cup O^{\cdot 4n^2} &\quad\mapsto\quad G_{8n,4n+4} \\ K'_{4n+4} \cup K'_{4n+4} \cup O^{\cup 2n+1} \cup O^{\cdot 2n(2n+1)} &\quad\mapsto\quad G_{8n+4,4n+4} \end{aligned}$$

*In each case, the initial and final links of the trail of $O$'s are disjoint pairs of vertices of $R$ (of the image). Also the $R$'s of the $G_{4n,4}$'s of the second packing are linked together and their image is a subset of the $R$ of $G_{8n,4n+4}$ which is disjoint from the image of any $O$.*

**Proof.** We shall first prove that triangles can be packed into $G = K_{n,n,n}$ with a suitably long trail. Write $G = E_A + E_B + E_C$, where $A = \{a_i : i \in \mathbb{Z}/n\mathbb{Z}\}$, $B = \{b_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ and $C = \{c_i : i \in \mathbb{Z}/n\mathbb{Z}\}$. The existence of a packing of $G$ with triangles follows from the existence of $n \times n$ latin squares. If the cells $(i, j)$ contain entries $k_{ij}$ then the corresponding triangles are given by $(a_i, b_j, c_{k_{ij}})$. Alternatively, a packing can be constructed from first principles as follows.

If $n$ is odd and for $i, j \in \mathbb{Z}/n\mathbb{Z}$, let $T_{ij}$ be the triangle $C(a_i, b_{i+j}, c_{i+2j})$. It is an easy exercise to show that the $T_{ij}$ pack $G$ exactly. If $n$ is even, use induction to pack $K_{n/2,n/2,n/2}$ with triangles and double up the vertices. The result is a packing of $K_{n,n,n}$ with octahedra. Since each octahedron can be packed with four triangles, we have exact packings of $K_{n,n,n}$ with triangles for all $n$.

We now find a long trail of triangles. Let $G[A, B] = K_{n,n}$ be the subgraph induced by the vertices of $A \cup B$. Each edge in this subgraph is contained in a unique triangle in the packing of $G$. There exists an Eulerian circuit of $K_{n,n}$ when $n$ even, and one of $K_{n,n}$ with a 1-factor removed when $n$ is odd, $n \geq 1$. Hence we have a circuit of length $n^2$ ($n$

*Table 3.* Packings of $K'_N$ used in Theorem 28.

| $N$ | $T_W$ | $T_S$ | $T_X$ | Isolated Components | $O$-trail |
|---|---|---|---|---|---|
| 16 | 13 | 36 | – | $G_{4,4}.G_{4,4} = W^{\cdot 2}$ | $O^{\cdot 4}$ |
| 20 | 29 | 56 | 28 | $K'_8 \cup K'_8 \cup O^{\cup 3}$ | $O^{\cdot 6}$ |
| 24 | 57 | 80 | 16 | $K'_8 \cup K'_8$ | $O^{\cdot 16}$ |
| 28 | 77 | 108 | – | $G_{8,4}.G_{8,4}$ | $O^{\cdot 16}$ |
| 32 | 123 | 140 | 60 | $K'_{12} \cup K'_{12} \cup O^{\cup 5}$ | $O^{\cdot 20}$ |
| 36 | 153 | 176 | 40 | $K'_{12} \cup K'_{12}$ | $O^{\cdot 36}$ |
| 40 | 209 | 216 | 72 | $G_{12,4}.G_{12,4}$ | $O^{\cdot 36}$ |
| 44 | 251 | 260 | 248 | $G_{24,20}$ | – |

even) or $n(n-1)$ ($n$ odd, $n > 1$) in this subgraph. By replacing the edges with triangles we get a circuit of triangles in $G$. By breaking this circuit at some vertex $v \in A$, say, we get a trail from some vertex of $C$ to another vertex of $C$. (Each triangle contains one vertex from each of $A$, $B$ and $C$ and if two triangles both meet $v$ then they must meet distinct vertices in $C$.)

Doubling up vertices, we get a packing of $O^{\cdot n^2}$ or $O^{\cup n} \cup O^{\cdot n(n-1)}$ into $K_{2n,2n,2n}$ with the trail of linked $O$'s starting and ending at disjoint pairs of vertices in $C$.

Now let $A$, $B$ and $C$ have $2n$ vertices, and let $S$ have 0, 2 or 4 vertices. We can decompose $G_{A\cup B,S\cup C} = K'_{A\cup B} + E_{S\cup C}$ as an edge disjoint union of $K_{A,B,C}$, $G_{A,S} = K'_A + E_S$ and $G_{B,S} = K'_B + E_S$. The result follows in the first two cases by replacing $n$ by $2n$ and setting $|S| = 0$ or $|S| = 4$. For $|S| = 0$ note that $G_{4n,0} = K'_{4n}$. The last case follows by replacing $n$ with $2n+1$ and setting $|S| = 2$, noting that $G_{4n+2,2} = K'_{4n+4}$. □

**Lemma 27.** *We can pack $G_{24,20}$ exactly with 124 pairs of linked triangles.*

**Proof.** Write $M = A \cup B \cup C$ where $A = \{a_i : 0 \le i \le 7\}$, $B = \{b_i : 0 \le i \le 7\}$ and $C = \{c_i : 0 \le i \le 7\}$ are disjoint sets of vertices of size 8. Decompose $K_A$ into seven 1-factors and combine each 1-factor with a set of edges $I_j = \{b_i c_{i+j} : 0 \le i \le 7\}$ with $1 \le j \le 7$ to get seven 1-factors of $K_{A\cup B\cup C} = K_M$. Cyclically permuting $A$, $B$ and $C$ now gives a total of 21 such 1-factors. The remaining edges of $K_M$ form 8 triangles $T_i = C(a_i, b_i, c_i)$. Discard one of the 1-factors and join the others, one to each of the 20 vertices of $R$. This gives a decomposition of $G_{24,20} = K'_M + E_R$ into 20 sets of 12 triangles linked at a vertex of $R$ and 8 disjoint triangles $T_i$. Form 8 pairs of linked triangles by taking $T_i$ and pairing it with the triangle containing the edge $b_i c_{i+1}$. All these triangles meet the same vertex $v \in R$. There are four remaining triangles meeting $v$ which we can pair up to form two pairs of linked triangles. For each of the other vertices in $R$, we have 12 linked triangles which can be paired up to form six pairs of linked triangles. We have now paired up all the triangles into 124 pairs which pack the whole of $G_{24,20}$ exactly. □

Finally, we give the proof of Theorem 1 for $N \equiv 0 \bmod 4$.

**Theorem 28.** *Assume the Induction Hypothesis with $N > 12$, $N \equiv 0 \bmod 4$. Then Theorem 1 holds for $N$.*

**Proof.** If $n_3+2n_6 \geq T_S$ where $T_S = \frac{1}{3}|E(G_{2n,r})|$, $r = r(N) = \frac{1}{2}(N-8)$ and $2n+r = N$, then we can pack the $C_3$'s and $C_6$'s into the trail of $T_S$ triangles in $G_{2n,r}$ given by Theorem 10. This trail ends at a vertex of $R$ so by linking a $C_3$ if necessary at this vertex we can pack a set of $C_3$'s and $C_6$'s exactly. The result will then follow from part 1 of Lemma 3 with $k \leq 1$.

Also, if $n_3 + 2n_6 \leq T_W$ where $T_W$ is given by Theorem 25 then we are done by Theorem 25. Since $T_W \geq T_S$ for all $N \geq 48$, the proof of the theorem is practically complete; all we have to do is to check the cases $N < 48$.

In most of these remaining cases we use the packings of Lemma 26. Pack $C_3$'s and $C_6$'s into the "isolated components" in Lemma 26. These isolated components are listed in Table 3. Isolated components $O = G_{4,2}$, $K'_8 = G_{6,2}$, $K'_{12} = G_{10,2}$ and $G_{12,4}$ can all be packed with trails of triangles by Theorem 10 with $N = 6$, 8, 12 and 14 respectively. In all these cases the number of triangles is even, so we can pack any $C_6$'s exactly (pack all the $C_6$'s before packing the $C_3$'s). For $N = 44$, pack $G_{24,20}$ with pairs of linked triangles using Lemma 27. These can also pack both $C_3$'s and $C_6$'s, and if we have enough we are done by induction from $N = 20$ ($G_{24,20}.K_{20} \subseteq K_{44}$). The total number of triangles $T_X$ needed to pack the isolated components is listed in Table 3. By Theorem 25 we can assume that $n_3 + 2n_6 > T_W$. In all cases where we can pack the isolated components exactly with triangles, $n_3 + 2n_6 > T_W \geq T_X$ so we have enough $C_3$'s and $C_6$'s to pack these exactly. Now pack the trails of octahedra with the remaining cycles and use Lemma 3 as in part 1 of the proof of Theorem 20. Note that the $O$-trails in Lemma 26 have initial and final links packed into *disjoint* pairs of vertices in $R$, so packing $O.T \cup O^{\cdot a-1}.P_{2,2}$ does not cause problems if we use the first $O$ from the trail $O^{\cdot a}$ for the $O.T$.

The only remaining cases now are $N = 16$ and $N = 28$.

1. The case $N = 16$.

Writing the cycle lengths as sums of 3's, 4's and 5's as described before Lemma 8, we can assume $2s_T + p_T \geq 8$ by Lemma 8. Pick a minimal set of cycles which involve $s$ 4's and $p$ 5's with $2s + p \geq 8$. From the above table we can assume $n_3 + 2n_6 > T_W = 13$. Consider the following cases separately.

(a) The case $0 \leq s \leq 2$.

We use the $W^{\cdot 2} \cup O^{\cdot 4} \mapsto G_{8,8}$ packing of Lemma 26. Pack $s$ of the $W$'s with the $[\underline{3}^a 3^a \underline{4}^{bd} 5^{bc} 5^{cd}]$ packing $(B)$ from the proof of Lemma 23. Pack the other $2 - s$ $W$'s with the $[5^a 5^b 5^c 5^d]$ packing of Lemma 23. All the squares in this packing are connected via $b$ and all the triangles are connected via $a$. Also, each square is connected to a pentagon and a connected sequence of two triangles within its own $W$. Since each cycle is written with at most one 5 or two 3's, we can pack every cycle of our subset that is written with at least one 4 entirely into the triangles, squares and pentagons of this packing. Since $2s+p \geq 8$, there are enough remaining cycles written with 5's to cover the remaining pentagons in the packing. However each of these cycles is precisely a pentagon (since a 5 cannot be written together with a 3 or another 5) so they pack entirely too. Since $n_3 + 2n_6 > 13$ there are enough $C_3$'s and $C_6$'s to pack any remaining triangles of the packing. At worst we may need to link a single $C_3$ to vertex $a$ to pack some $C_6$. We therefore have an exact packing of some of our original cycles into $W^{\cdot 2}$ or $W^{\cdot 2}.C_3$.

(b) The case $s \geq 3$ and $p \geq 2$.
First assume $s = 3$ and $p = 2$. Pack one $W$ using the $[\underline{3}^a 3^a \underline{4}^{bd} 5^{bc} 5^{cd}]$ packing $(B)$ and the other $W$ using the $[3^a 3^a \underline{3}^b \underline{3}^c 4^{bd} 4^{cd}]$ packing $(A)$ from the proof of Lemma 23. Every square and pentagon is connected to every other square and pentagon in this packing. The triangles can be paired up into connected sequences of length 2 with one end meeting $a$ and the other meeting a square. Hence we can pack our subset of cycles entirely into this packing with any remaining triangles all linked to $a$ in disjoint connected sequences, each of length at most two. Packing the remaining triangles with $C_3$'s and $C_6$'s as in (a) allows us to exactly pack the whole of $W^{\cdot 2}$ or $W^{\cdot 2}.C_3$ with some of our original cycles.

Now assume that either $s > 3$ or $p > 2$. No cycle in the minimal subset can be a pentagon since we could remove it and still have $2s + p \geq 8$. Hence each cycle written with a 5 contributes at least three to the sum $2s + p$. By minimality there cannot be more than three such cycles, so $p \leq 3$. If we remove one of the cycles written with a 5 then $s \leq 3$ and $p \leq 2$ for the remaining cycles. Hence we can split a cycle written with a 5 into two parts so that one part and the remaining cycles of the subset have $s = 3$ and $p = 2$. We can pack these as above and attach the other part to a suitable vertex in $\{b, c, d\}$ so as to connect it with the part that was packed. In this case we have packed $W^{\cdot 2}.(C_r)$ or $W^{\cdot 2}.(C_r \cup C_3)$ with the $C_r$ linked at one of $\{b, c, d\}$ and the $C_3$ linked at $a$.

(c) The case $p \leq 1$.
Hence $s \geq 4$. First assume $s = 4$ and $p = 0$. Pack both $W$'s using $[3^a 3^a \underline{3}^b \underline{3}^c 4^{bd} 4^{cd}]$. All the squares are connected via $d$, so we can pack the cycles of our subset into this and pack $C_3$'s and $C_6$'s into any remaining triangles as before. If $s > 4$ or $p > 0$ then we can split one of the cycles as in (b). At worst we will need to attach one cycle to one of $\{b, c, d\}$, and possibly a $C_3$ to $a$ to pack the original cycles fully.

In all cases we have an exact packing of some of the original cycles into $W^{\cdot 2}.(\cup C_{l_i})$ where there are at most two cycles linked to distinct link vertices of $W^{\cdot 2}$ and if there are two such cycles then one of them is a $C_3$. Pack the longer one of these and the remaining cycles into the $O$-trail using Theorem 18. The total length of these cycles is at least $|E(K'_{16})| - 2 - 40 - 3 = 12a + 17$ with $a = 4$, so this succeeds. Now continue with the argument used in the proof of Theorem 25 to get the desired packing.

2. The case $N = 28$.
From the table above we may assume $77 = T_W < n_3 + 2n_6 < T_S = 108$. First assume $n_5 \geq 2$. We can pack $G_{8,4}$ with a $C_5$ and a circular connected sequence of triangles by Lemma 7. Hence we can pack $G_{8,4}.G_{8,4}$ with two $C_5$'s and a connected sequence of 34 triangles, which in turn can be packed exactly with $C_3$'s and $C_6$'s. We are now done as before by packing the $O$-trail and using Lemma 3 as in Theorem 20. Assume now that $n_5 \leq 1$. From Lemma 7, we can pack $G_{8,4}.G_{8,4}$ with four squares and a trail of 32 triangles. We can make the four squares form a connected sequence meeting three link vertices, or two connected pairs meeting all four link vertices (by suitable choice of which link vertices the pair of squares in each $G_{8,4}$ meet). Pack the trail of triangles with $C_3$'s and $C_6$'s as above. Write the cycles of length 4 or $\geq 7$ as sums of 3's, 4's and 5's as described before Lemma 8. Since each 4 can be written together with at most one 5 or two 3's, the total length of such cycles is at most $10 s_T$. However, since

$n_3 + 2n_6 < T_S$, the total length of cycles of lengths four or at least seven is at least $|E(K'_{28})| - 2 - 5n_5 - 3(T_S - 1) \geq 36$. Hence $s_T \geq 4$. Pack a minimal set of cycles written with at least four 4's into the packing $G_{8,4}.G_{8,4}$ described above. To do this we may need to attach up to four additional cycles $C_{l_i}$ to the link $R$ of $G_{8,4}.G_{8,4}$. By considering separately the cases when each of the packed cycles is written with one 4, or one is written with more than one 4, we can ensure that the linked cycles $C_{l_i}$ meet *distinct* link vertices. At most one of these cycles $C_{l_i}$ need contain a 4, so removing the longest of these gives a total length of additional cycles of at most $3 \times 6 = 18$. Pack the longest $C_{l_i}$ and all remaining cycles into the $O$-trail using Theorem 18. This succeeds since these cycles have total length at least $12a + |E(K'_{12})| - 2 - 18 = 12a + 30$. Since there are at most two $C_5$'s, the last form in Theorem 18 does not occur and we will only need to attach at most one cycle $P_{c,d}$ to the final link of this trail. We are now done by the same argument as in Theorem 25. □

Theorem 1 now follows by induction of $N$ using Lemmas 6 and 8 for the cases $N \leq 12$ and Theorems 20, 22 and 28 for the cases $N > 12$.

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