Continuum Percolation with Steps in the Square or the Disc

Paul Balister[*‡] Béla Bollobás[*†‡§] Mark Walters[*†‡§]

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Abstract

In 1961 Gilbert defined a model of continuum percolation in which points are placed in the plane according to a Poisson process of density one, and two are joined if one lies within a disc of area A about the other. We prove some good bounds on the critical area A_c for percolation in this model.

The proof is in two parts: first we give a rigorous reduction of the problem to a finite problem, and then we solve this problem using Monte-Carlo methods. We prove that, with 99.99% confidence, the critical area lies between 4.508 and 4.515. For the corresponding problem with the disc replaced by the square we prove, again with 99.99% confidence, that the critical area lies between 4.392 and 4.398.

1 Introduction

In this paper we consider some continuous percolation questions. The general question we shall consider is the following: Let Λ be a Poisson process of density λ in the plane and let A be a symmetric open set. We form a graph $G_A(\Lambda)$ with vertex set Λ by joining two points a, b of Λ if $b - a \in A$ (since A is symmetric, the relationship is symmetric). We wish to know for what values of λ percolation occurs (i.e., the graph $G_A(\Lambda)$ has an infinite component with positive probability). Standard results show that there is a critical density λ_c (possibly zero) such that percolation occurs if $\lambda > \lambda_c$ and does not occur if $\lambda < \lambda_c$. Since

[*]Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA
[†]Trinity College, Cambridge CB2 1TQ, UK
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replacing A by kA multiplies the critical density by $1/k^2$, we shall assume that A has unit area. Thus λ_c is exactly the average degree of a vertex at the critical point, and is also equal to the 'critical area' A_c, defined by taking the density to be one and scaling the region A. Our aim is to prove good bounds on λ_c for two of the most natural choices for A, namely, the square and the disc (i.e., the unit area balls in the l_∞ and l_2 norms respectively).

The general method we use is the following. We reduce the continuous model to a bond percolation on the square lattice. The bonds will *not* be independent in this model but there will be sufficient independence to enable us to prove that percolation occurs if the probability that a bond occurs is high enough. This reduces the problem to that of evaluating a very large but finite dimensional numerical integral. This is impractical to evaluate rigorously so we use Monte-Carlo methods.

These models have been widely studied since their introduction by Gilbert [12]. Various rigorous bounds have been proved but they are very weak: the upper and lower bounds are a factor of about four apart. For an overview of these results see Grimmett [13] and particularly Meester and Roy [20]. Simulation methods have been used extensively; for the square see Dubson and Garland [7], Alon, Drory and Balberg [1], King [17], Garboczi, Thorpe, DeVries and Day [11], and most recently Baker, Paul, Sreenivasan and Stanley [3]; for the circle see Roberts [25], Domb [6], Pike and Seager [21], Fremlin [10], Haan and Zwanzig [16], Gawlinski and Stanley [14], Rosso [26], Lorenz, Orgzall and Heuer [19], Quintanilla and Torquato [23] and most recently Quintanilla, Torquato and Ziff [24]; these are significantly different from our results in that they model on the situation on a finite grid and assume that this implies bounds for the infinite grid. Indeed it should be noted that more recent results are often not within the error margins given by earlier papers. Our bounds, on the other hand, are completely rigorous — the only reason we are using Monte-Carlo methods is because we want to evaluate a complicated numerical integral. This does mean that our results only hold with a certain confidence level: in this case 99.99%. For some results in a similar vein see Bollobás and Stacey [5] where the authors prove some bounds on the critical probability for oriented site percolation.

The methods we use are also able to prove bounds on the critical probability of site percolation on the square lattice.

The layout of the paper is as follows. In the next section we will define a dependent bond percolation model and prove that, under certain conditions, it does percolate. In the subsequent two sections we reduce our problem to this bond percolation model. In the fifth section we describe how we can apply our method to site percolation on the square lattice. Then we describe the results from our Monte-Carlo simulation. In the final section we prove that a more general dependent bond percolation model percolates.

2 A Dependent Bond Model

We consider percolation on the standard lattice $\mathbb{Z}^2$. We suppose that each edge (or *bond*) is open with probability at least p. However we do not insist that the edges are independent; we only insist that each edge is independent of all edges disjoint from it. More formally we say a bond percolation model is 1-*dependent* if any set S of bonds is independent of any collection T of bonds with the distance from S to T at least one. Percolation of this type was studied by Durrett [8], Andjel [2], Pisztora [22], Liggett, Schonmann and Stacey [18], and others.

One such model is the following: label each site with an independent 0–1 Bernoulli random variable with mean r. A bond is declared open if its endpoints have the same value. Then this is 1-dependent. However if $0.5 < r < p_{\mathrm{site}}$ where p_{site} is the critical probability for site percolation on the square lattice ($p_{\mathrm{site}} \approx 0.593$) then percolation does not occur. However, the probability that any particular bond is open is $r^2 + (1-r)^2 > 0.5$. This shows that we can have the probability that a bond is open strictly greater than one half without percolation occurring. Let p_0 be the minimal real number such that in any 1-dependent model, percolation does occur if the probability that an edge is open is greater than p_0. By the example above $p_0 > 0.5$. Our aim is to prove a good upper bound for p_0; this will be done in Theorem 2. First, we show that $p_0 < 1$. This is an immediate consequence of Theorem 0.0 of [18], but as we need it in our proof of Theorem 2, we prove it here.

Lemma 1. *The critical probability p_0 defined above is strictly less than* 1.

Proof. We use a simple boundary path counting argument. Let C be the set of vertices in the component of the origin. Assume C is finite. Let E be the infinite component of $\mathbb{Z}^2 \setminus C$, and B the set of edges joining C to E. Then C defines a simple closed boundary path P in the dual lattice $(\mathbb{Z}^2)^* = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ of $\mathbb{Z}^2$ by the property that an edge of $(\mathbb{Z}^2)^*$ is in P if and only if the unique edge crossing it in the original lattice is in B. The edges of the set B must all be closed, and the subgraph of the lattice induced by B is bipartite with maximum degree 4. Hence there is a proper edge coloring of B with 4 colors, and so there exists an independent set in B of size at least $|B|/4$. (If C is not a single point then this can be improved to $|B|/3$.) Since these edges are independent, the probability of the edges in B all being closed is at most $q^{|B|/4}$ where $q = 1-p$. However $|B|$ is just the length of the path P. The number of possible sets B of size l is the number of possible paths P of length l and this is clearly bounded above by $l3^l$ (the path must cross the positive x-axis within l of the origin, and at each step of the path there are at most 3 possible directions that can be taken). Hence the probability that C is finite (and so B exists) is bounded above by $\sum_{l=4}^{\infty} l3^l q^{l/4}$. However, for sufficiently small q, this is less than 1 and so C is infinite with positive probability. $\square$

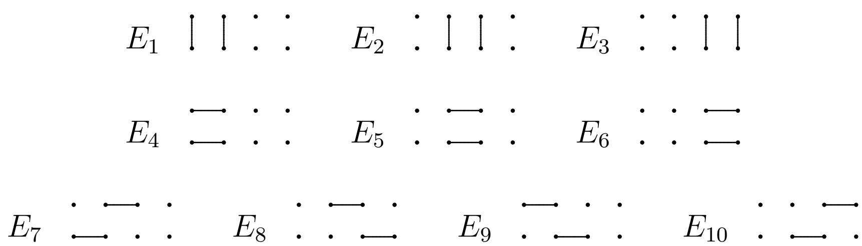


Figure 1: The sets E_i.

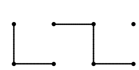
Theorem 2. *A strict upper bound for p_0 is 0.8639, i.e., in any 1-dependent bond percolation on $\mathbb{Z}^2$ in which every bond is open with probability at least 0.8639, the origin is in an infinite component with positive probability.*

Proof. We illustrate the method first with a simpler proof that we can take $p_0 = 0.9$. The idea is to use renormalization. Consider the lattice $(2\mathbb{Z})^2$. The vertex u of this lattice will correspond to the 2×2 square $S_u = \{u + (0, 0), u + (0, 1), u + (1, 0), u + (1, 1)\}$ of the original lattice, and a bond uv will correspond to the rectangle $S_u \cup S_v$. We declare this bond to be open, if in the graph induced by $S_u \cup S_v$, there is an open component containing at least 3 vertices from S_u and at least 3 vertices from S_v. It is clear that if there is an infinite component in $(2\mathbb{Z})^2$ then there is an infinite component in the original lattice, and that for any collection of bonds S, the sigma-algebra generated by S is independent of the sigma-algebra generated by the edges not incident to any edge of S. Assume bonds are closed in the original lattice with probability at most $q = 1 - p$. We shall show that each bond in $(2\mathbb{Z})^2$ is closed with probability at most $q' \leq 10q^2$.

First note that we can assume without loss of generality that every bond is closed with probability *exactly* q. To see this, delete each bond independently with a suitable probability (possibly depending on the bond), so that the bond remains open with probability p. This new model is still 1-dependent and no more likely to percolate than the original.

Draw $S_u \cup S_v$ horizontally, and consider the 10 sets of edges shown in Figure 1. The probability that both edges in any E_i are closed is q^2 (the two edges in each E_i are independent since they are disjoint). So with probability at least $1 - 10q^2$, none of the sets E_i consist of two closed edges. We shall show that in this case the bond uv is open.

By considering the sets E_1–E_6, we see that the graphs induced by S_u, S_v, and the 'middle' square all have two incident open edges, and hence components of size at least 3. If these components did not join up then we would have a configuration such as



However, in these cases one of the bonds of the sets E_7–E_{10} would join the components (E_9 in the example above).

Now inductively renormalize the lattice to obtain a sequence of models on $(2^i\mathbb{Z})^2$ in which the bonds are closed with probability q_i and $q_{i+1} \leq 10q_i^2$, $q_0 = q$. If $q < 0.1$ then $q_i \to 0$ and so eventually we can use Lemma 1 to show the existence of an infinite cluster.

There are two possible improvements to this method. One is to note that we can get slightly better bounds on the probability that all the E_i are closed. For instance, the probability that at least one of E_7 and E_8 is closed is $q(1-p^2) = q^2(2-q)$ since all three edges in $E_7 \cup E_8$ are independent. This reduces the quantity $10q^2$ slightly and improves the result. Another improvement is to work with 3×3 squares in the renormalization. Here we declare a bond uv of $(3\mathbb{Z})^2$ to be open if, in the graph induced by $S_u \cup S_v$, there is an open component containing at least 5 vertices from S_u and at least 5 vertices from S_v. As before we find sets E_i such that, whenever the edge uv is closed, at least one of the E_i consists entirely of closed edges. The sets E_i that we use are shown in the first column of Figure 2. In total there are 57 sets E_i with 3 independent bonds and 18 sets with 4 independent bonds. Thus we obtain an infinite cluster when $57q^3 + 18q^4 < q$. This gives a value of p_0 just below 0.8702. However, if we have two or more such sets whose union is a vertex disjoint collection of edges, then we can calculate the probability that at least one of these sets is absent exactly (since the edges are all independent). Figure 2 shows the collections we use, the union of the sets in each collection, and the probability that at least one set is absent (as a polynomial in q) from each collection. We see that there is an infinite cluster provided that $57q^3 - 21q^4 - 11q^5 - 4q^6 + 3q^7 < q$ which is satisfied if $p \geq 0.8639$. □

It would be of interest to give significantly better bounds for p_0; unfortunately we cannot even hazard a guess as to the value of p_0.

To conclude this section, let us make some remarks we shall need later (in the proof of Theorem 5). First, note that with probability at least $1 - 12q_i$ there is a circuit surrounding the origin in the $(2^i\mathbb{Z})^2$-process that corresponds to a circuit in the original process surrounding the square $[-2^i, 2^i]^2$. Letting $i \to \infty$ we see that with probability 1, any point is surrounded by arbitrarily large circuits. In particular, this implies the infinite connected component is unique (pick points from 2 infinite components, both are surrounded by the same large circuit, which must then lie in both components). Also there are no infinite components in the dual process (i.e., the corresponding process in the dual lattice).

3 Upper Bounds

First we give the reduction to the bond percolation on the square lattice. For any region S of the plane let $G_A(\Lambda|_S)$ denote the graph with vertex set $\Lambda \cap S$ formed by joining two

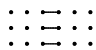
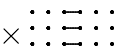
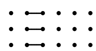
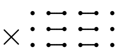
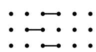
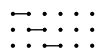
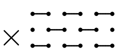
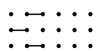
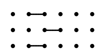
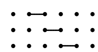
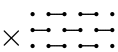
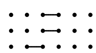
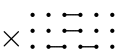
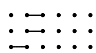
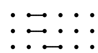
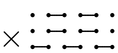
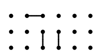
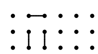
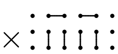
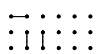
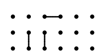
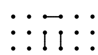
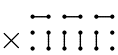
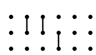
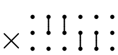
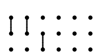
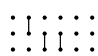
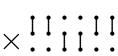
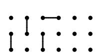
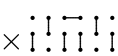
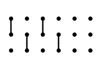
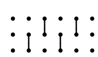
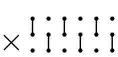
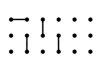
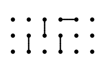
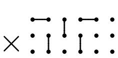
Sets E_i	Union	q^3	q^4	q^5	q^6	q^7
1× 	1× 	1				
2× 	1× 	2			−1	
2×  4× 	1× 	6	−5	−2	2	
2×  2×  2× 	1× 	6	−6	1	−1	1
4× 	2× 	4	−2			
4×  4× 	2× 	8	−4	−2	−2	2
4×  4× 	2× 	8	−6			
4×  4×  2× 	2× 	10	−8		−2	2
4× 	2× 	4	−2			
4×  4× 	2× 	8	−6			
4× 	2× 		4			−2
4×  2× 	2× 		6	−4		
4×  4× 	4× 		8	−4		
Total		57	−21	−11	−4	3

Figure 2: Sets E_i for 3×3 renormalization. ($n\times$ indicates there are n patterns obtained from the given one by horizontal and/or vertical reflections).

vertices x and y if $x - y \in A$. Cover the plane by a grid of squares of side length R. Associate with each square a vertex of the lattice in the natural way. Two neighboring vertices with corresponding squares S, T in the lattice are joined if in the graph $G_A(\Lambda|_{S \cup T})$ the component of $G_A(\Lambda|_S)$ with the maximal number of vertices is joined to the component of $G_A(\Lambda|_T)$ with the maximal number of vertices. (Note that if either of these components is not unique then we say that S and T are *not* joined.) This gives us a bond percolation model. Moreover if percolation occurs in this model then it occurred in the continuous model. Finally observe that this model is 1-dependent. Thus, by stochastic domination, we have the following result.

Theorem 3. *Suppose that we have a process as described above. Let S and T be two adjacent squares. Suppose that the maximal component of S is joined in $S \cup T$ to the maximal component of T with probability at least* 0.8639. *Then the process percolates.*

We shall show that the conditions of this theorem are satisfied for certain shapes A and values λ. We do this using a Monte-Carlo simulation; see Section 6 for further details.

4 Lower Bounds

By a variation on the above method we prove some lower bounds. Let the *connection process* $\mathcal{P} \subset \mathbb{R}^2$ be the set of points of the Poisson process Λ together with all line segments joining neighboring points in the process, i.e., points x and y with $x - y \in A$. Let $\mathcal{P}^c$ denote the complement of $\mathcal{P}$ in $\mathbb{R}^2$.

As before cover the plane with a grid of squares of side length R. Let S and T be two neighboring squares. We wish to say that S and T are joined if some configuration occurs in the complement of the process, $\mathcal{P}^c$. However this event must not depend on points outside of the rectangle $S \cup T$.

Let W be the rectangle $S \cup T$. Let W' be the rectangle $\{x \in W : x + A \subset W\}$. Then the connection process inside W' depends only upon the point process inside $W = S \cup T$. Define S' and T' similarly.

Suppose S and T are neighbors with T to the right of S. We define the two squares S, T to be joined if the following conditions hold (see Figure 3 for a picture).

1. There is a path in $\mathcal{P}^c$ from the left to the right of the rectangle W'.

2. There is a path in $\mathcal{P}^c$ from the top to the bottom of the square S'.

(All paths must lie entirely within the rectangle concerned.) Similarly define joined when S and T are neighbors with T above S. We will say that a rectangle $W = S \cup T$ is *open* if S is joined to T.

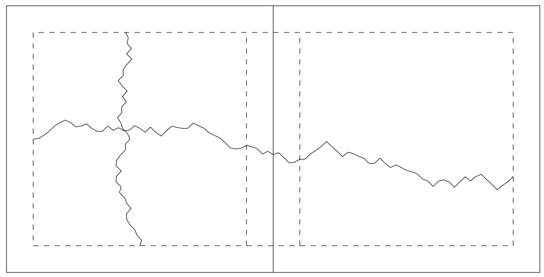


Figure 3: The two squares in the figure are joined in the model for the lower bound.

Lemma 4. *Suppose we have two such rectangles W and V that overlap (in any configuration). Suppose further that both W and V are open. Let w and v be subsets of $W \cap \mathcal{P}^c$ and $V \cap \mathcal{P}^c$ respectively demonstrating openness. Then w and v intersect.*

Proof. Suppose that the rectangles overlap in a square S. Then, whatever the orientation of the rectangles W and V, there exists a path from the top to the bottom of S' in w or in v and a path from the left to the right of S' in the other. These must intersect. □

Theorem 5. *Suppose that we have a process as described above. Let S and T be two adjacent squares. Suppose that the probability that S and T are joined is at least 0.8639. Then the process does not percolate.*

Proof. Consider the process defined on the square lattice in the natural way. Then a particular edge occurs with probability at least 0.8639 and, moreover, the model is 1-dependent. Thus by the remarks following Theorem 2 this process contains arbitrarily large circuits about any point. These edges contain the configuration defined above in $\mathcal{P}^c$. Since these configurations in neighboring rectangles overlap we have arbitrarily large circuits in $\mathcal{P}^c$ about any point. Thus the original process $\mathcal{P}$ does not percolate. □

We now prove a criterion for a rectangle being open that is easy to check.

Lemma 6. *Suppose that W is a rectangle and that A is convex. Suppose that there does not exist a path from the left to the right of W' in $\mathcal{P}^c \cap W'$. Then there exist points x and y such that $x + A$ intersects the top of W and $y + A$ intersects the bottom of W with x and y joined in the original process.*

Proof. Since there is no path in $\mathcal{P}^c$ from the left to right side of W' there is a topological path in $\mathcal{P}$ from the top to the bottom of W' (since $\mathcal{P}$ is a finite union of line segments the topology is simple). We may assume that the endpoints of this path are points in the original process.

However, since the shape A is convex, any two points x and y are joined if and only if $x + \frac{1}{2}A$ and $y + \frac{1}{2}A$ intersect. Thus, if x and y are joined in the original process the entire line segment xy is contained in $(x + \frac{1}{2}A) \cup (y + \frac{1}{2}A)$. Thus, if an edge xy intersects an edge uv then at least one of the pairs xu, yu, xv, yv form an edge. Therefore if we have a topological path in $\mathcal{P}$ then we can insist that it is a path in the graph sense.

Therefore there exists a graph path in the original process crossing from the top of W' to the bottom of W'. Letting x be the top endpoint of this path and y the bottom endpoint the result follows. $\square$

From the lemma we see that all we need to check to show that a rectangle is open is the non-existence of two paths in the original process.

5 Site Percolation on the Square Lattice

Site percolation on the square lattice is very similar to a discrete version of our problem. We choose points uniformly at random in the lattice (equivalent to the Poisson process in the continuous model) and join two points if their difference lies in the set $A = \{x : \|x\|_1 \leq 1\}$. Indeed, since our method actually uses a discrete version of the problem (see next section), this problem can be simulated directly by our program. The results are included in Section 7.

6 Random Methodology

We ran the simulations for three models: the square, the circle and the lattice site percolation. We used the random number generators provided by the GNU Scientific Library [15]. For our main results we used the MT19937 generator. We also verified that the Tausworthe generator gave the same results on some smaller grids. They are both high quality generators with very long periods (approximately 10^{6000} for the MT19937).

We give the results of our simulations in the next section. We are aiming to show that the probability of an event is at least 0.8639. Thus we need a bound on the probability that we would see these results even when the probability of the event occurring is less than 0.8639. We could use a bound on the tail of the binomial distribution (for example the Chernoff bound) but it is simpler and more accurate to sum the binomial series directly. It turns out that simulating a bond 200 times is about optimal; if we need more simulations than that we do better to run the simulation on a larger rectangle. If we have at most 10 failures from the 200 attempts then we have 99.99% confidence that the probability the event occurs is at least 0.8639. If we have at most 13 failures we still have almost 99.9% confidence.

There is one small technicality: for practical reasons our program approximates the continuous process by a discrete version. The approximations we use differ for the upper bounds and lower bounds.

First we give the approximation for upper bounds. We cover the plane with a very fine square mesh. We identify this with a site percolation on $\mathbb{Z}^2$. We put a site in if its corresponding square in the mesh contains a point. Then we join two sites x and y with corresponding squares S and T if every pair of points $s \in S$ and $t \in T$ are joined in the continuous process (i.e., $s - t \in A$). Then percolation in this process implies percolation in the original process.

For lower bounds our construction is almost the same, but this time we join two sites x and y with corresponding squares S and T if there exist points $s \in S$ and $t \in T$ that are joined in the continuous process (i.e., $s - t \in A$). Then percolation in the continuous process implies percolation in this process.

Thus in either case the bound for the discrete version gives the bound for the continuous version. The results we give in the next section have been corrected for this. The grid mesh we used was $2^{28} > 10^8$ times smaller than the radius of the shape A so this correction was very small.

Having written our program we chose the grid size based on reasonable run time; then we did some trials on smaller grids and heuristically determined reasonable densities to try on the full grid. This meant that we did not require multiple attempts at our final grid size which would have wasted computer time and invalidated our confidence levels.

7 Results

Recall that all of our results are normalized for a region A of unit area. Thus the density is precisely the average degree in the graph. We begin with the square. We have lower and upper bounds of 4.392 and 4.398 respectively with 99.99% confidence. Table 1 gives some previously published estimates given by computer experiments.

Direct comparison between our bounds and the above estimates is difficult for several reasons. The above results are usually stated as an estimate together with an error margin; the error margin is meant to be one standard deviation. The above estimates are including only a single standard deviation and thus only hold with 68% confidence. Our bounds hold with 99.99% confidence; to obtain this level of confidence we would need about four

Lower	Upper	Year
4.388	4.396	2002 [3]
4.374	4.447	1990 [17]
3.98	4.44	1991 [11]
4.489	4.509	1990 [1]
3.66	3.93	1985 [7]

Table 1: Some experimental bounds for the square of unit area.

Lower	Upper	Year
4.51218	4.51228	2000 [24]
4.512	4.513	1999 [23]
4.502	4.524	1993 [19]
4.509	4.521	1989 [26]
4.48	4.53	1981 [14]
4.55	4.63	1977 [16]
4.38	4.42	1976 [10]
4.47	4.54	1974 [21]
4.43		1972 [6]
3.87		1967 [25]
3.2		1961 [12]

Table 2: Some experimental bounds for the disc of unit area.

standard deviations which would multiply the width of the above intervals by four. Secondly the above estimates are just that: they are estimated extrapolations from finite grids. Our results on the other hand require no extrapolation: they are true unless we are exceptionally unfortunate in the numbers produced by our random number generator.

In this case we have reduced the error margins significantly and with a much more rigorous method.

The circle has been more intensively studied than the square and the estimates are correspondingly better. Our lower and upper bounds are 4.508 and 4.515 respectively. Table 2 shows some previously published estimates.

We are still able to better all but the most recent results and again with a much more rigorous method.

Finally we turn to site percolation on the square lattice. This is one of the most extensively studied bounds in percolation so we do not expect to be able to improve on previously known results. It is interesting to see what our more rigorous method gives in this case. Our lower and upper bounds are 0.5919 and 0.5935. These agree with the best known estimate by Ziff [28] of 0.592746 but are much weaker. Again, however, it should be noted that our bound does not involve any extrapolation. Truly rigorous bounds are much weaker: Ermakov and van den Berg 0.556 [9] and Wierman 0.679 [27].

8 Extensions of Dependent Percolation

A bond percolation model is d-dependent if any two sets of bonds S and T such that the l_∞-distance between them is at least d are independent. It is immediate that the analogue of Lemma 1 holds for d-dependent percolation: percolation occurs provided the probability that a bond is open is high enough. More importantly, we prove asymptotic bounds on how high this probability must be. To avoid using the Monte-Carlo method, we first prove the following theorem.

Theorem 7. *The independent site percolation process with probability p on $\mathbb{Z}^2$ where sites are joined if they are at l_∞-distance $\leq d$ percolates when $p \geq \frac{2.73}{d(d+1)}$, or equivalently when the expected number of open neighbors of a point is at least* 10.92.

Proof. Consider a new process: site percolation on $\Lambda = (r\mathbb{Z})^2$, where $r = \lceil \frac{d}{2} \rceil$ and a point is joined to all eight of its neighbours at distance r (in the l_∞ norm). A site u in Λ is defined to be open if any element of $u + [0, r-1]^2$ is open in the original process. If u and v are neighbours in Λ then every point in the square corresponding to u is joined to every point in the square corresponding to v. Thus if we have percolation in this new process we have percolation in the original process. However, the new process is isomorphic to 8-neighbour site percolation on the square lattice. Since normal 4-neighbour site percolation is dual to 8-neighbour site percolation, a lower bound for the critical probability for 4-neighbour site percolation provides an upper bound for the critical probability in 8-neighbour site percolation. A rigorous bound of 0.556 for this was proved by Ermakov and van den Berg [9]. Thus, if $q = 1 - p$ is the probability a site is closed in the original process, we have percolation provided that

$$1 - q^{r^2} > 1 - p_{\mathrm{site}} \geq 1 - 0.556$$

which is true if $p > 1 - (0.556)^{1/r^2}$ or, equivalently

$$d(d+1)p > d(d+1)(1 - (0.556)^{1/r^2}).$$

The right hand side takes it maximum when $d = 4$ with value 2.73. Thus $p > \frac{2.73}{d(d+1)}$ implies that we have percolation. $\square$

Theorem 8. *Suppose that X is a d-dependent bond percolation model in which each edge is open with probability $p = 1 - q$. Then if $q \leq \frac{0.1361}{d}$ percolation occurs. Moreover, there exists such a model with $q \leq \frac{10.92}{d}$ for which percolation does not occur.*

Note the $\frac{10.92}{d}$ can be reduced to $\frac{4.40}{d}$ if our Monte-Carlo results are used and d is sufficiently large.

Proof. For the first part, consider just the bonds lying in the lines $x = kd$ and $y = kd$, $k \in \mathbb{Z}$. Construct a bond percolation on $(d\mathbb{Z})^2$ by declaring a bond uv with $\|u - v\|_1 = d$ to be open if all d edges in the original process on the straight line from u to v are open. The probability of such an edge being closed is at most dq, and the bonds are independent of any set of non-incident bonds. Hence, by Theorem 2, if $dq \leq 1 - 0.8639 = 0.1361$ the new process percolated. However, percolation in the $(d\mathbb{Z})^2$-process implies percolation in the original.

For the second part, construct a bond percolation process on $\mathbb{Z}^2$ in the following manner. Let ξ_u be a collection of independent Bernoulli random variables, defined for each $u \in \mathbb{Z}^2$, each equal to 1 with probability $\frac{10.92}{4d(d-1)}$. For each $u \in \mathbb{Z}^2$, if $\xi_u = 1$ declare all the edges joining the boundary of $u + [0, d]^2$ to the boundary of $u + [1, d-1]^2$ to be closed. All edges not declared closed will be open. There are $4(d-1)$ such edges, so the probability of an edge being closed is at most $\frac{10.92}{d}$. Moreover, the edges are independent at distance $\geq d$. If we consider the dual process on $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ we see that this percolates iff there is percolation in the site process on $\mathbb{Z}^2$ given by ξ_u where sites are joined if they are at l_∞-distance at most $d - 1$. By the previous theorem we know that this process percolates, so the original process does not. $\square$

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